*The world of polyhedra**

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Geometrical solids can become the subject of a fascinating study. Not everyone, of course, will want to make the attempt to understand all the theoretical mathematics involved in discovering and classifying those solids known as uniform polyhedra and their stellated forms. But everyone surely can appreciate the beauty and symmetry of these solids, whose history is as ancient as Plato, Archimedes, and Euclid and as modern as Coxeter, today a well-known professor of mathematics at the University of Toronto.

It is a fact well known to anyone acquainted with mathematics that the Thirteen Books of Euclid's *Elements*, for centuries the only textbook of geometry, begins with a proposition describing how to construct an equilateral triangle and ends with five propositions on the so-called five Platonic solids. These five solids are the tetrahedron, octahedron, icosahedron, hexahedron (cube), and dodecahedron. The last proposition in Euclid's *Elements* states that no other solids with congruent regular polygons as faces are possible, and the proof of this proposition brings this famous work to a close.

As history moved into the modern era, interest in polyhedra revived in the same way as interest in many other fields of human knowledge. The astronomer Johannes Kepler (1571–1630) was fascinated by the relationships among the five Platonic solids, and he even tried to work out a mathematical relationship connecting these solids with the distances between the sun and the planets known in his day. Some data in his theory approached reality closely enough to appear convincing, but further investigation eventually made his theory hopelessly inadequate. But Kepler's study of polyhedra did lead him to discover two new solids which, in an extended way, can be called regular, and thus he added something to the world of polyhedra. He came to his discovery when he noticed that producing the sides of an equilateral triangle or of a square will not lead to new polygons, but producing the sides of a pentagon will lead to a star polygon. Therefore, by taking the dodecahedron and producing each of its edges, a new solid will be formed whose faces can be thought of as a set of twelve interlocking or intersecting star polygons, meeting five at each vertex with twelve vertices in all. This new solid was later named the small stellated dodecahedron. Kepler also discovered the great stellated dodecahedron, which also has twelve star polygons as faces, meeting three at each vertex with twenty vertices in all.

Almost two hundred years later, in 1809, the French mathematician Louis

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Poinsot (1777–1859) discovered two more regular polyhedra, the great dodecahedron, having twelve mutually intersecting pentagons as faces, and the great icosahedron, with twenty mutually intersecting equilateral triangles as faces. Augustin-Louis Cauchy (1789–1857), another famous mathematician of this time, showed that these arise by stellating the dodecahedron and the icosahedron respectively, and he also proved that the four Kepler-Poinsot solids exhaust the list of finite regular polyhedra. In other words, to Plato's original five, modern mathematics, in an extended way, has added four more, bringing the total to nine regular solids.

In the case of the icosahedron, the process of stellation leads to polyhedra that are not regular in the sense used above but which nevertheless possess great beauty and symmetry. The process of stellation may be described as follows. Any one of the five Platonic solids may be imagined as resting on a horizontal plane, say a table top. The plane of the table top may thus be imagined to be the extension of the base plane of the solid. Next, each face of the solid may similarly be imagined with its own extended facial plane. Some planes turn out to be parallel. Those which are not parallel will intersect. The four intersecting planes of the tetrahedron enclose only the tetrahedron itself. The same thing happens with the cube; the six planes enclose only the cube itself. The eight planes of the octahedron lead to something more interesting. Besides the original octahedron, there will be eight small tetrahedra, each with one of its faces in common with one of the faces of the octahedron. Thus the octahedron leads to one stellated form, which Kepler called the *stella octangula*, the eight-pointed star. It may also be thought of as a compound of two interpenetrating tetrahedra. It also has the property that its eight points or vertices coincide with the eight vertices of a cube; its edges are diagonals of the square faces of a cube.

With the dodecahedron, this method of stellating a solid by producing its facial planes leads to the formation of three distinct types of cells enclosed by the intersecting planes. Besides the dodecahedron itself, there will be twelve pentagonal pyramids. These convert the dodecahedron into the small stellated dodecahedron. Then there will also be thirty sphenoids, or wedge-shaped pieces, which convert the small stellated dodecahedron into the great dodecahedron. Finally, there will be twenty triangular dipyramids which convert the great dodecahedron into the great stellated dodecahedron. Thus, the dodecahedron leads to three stellated forms. As noted before, two of these were discovered by Kepler, the third by Poinsot.

What we may call the exterior parts of these stellated forms may easily be found by drawing what is called a stellation pattern. For the octahedron, this is a triangle within a triangle, the inner one with its vertices at the midpoints of the sides of the outer one. (See Fig. 1.) For the dodecahedron, a star polygon within a star polygon will give the pattern (Fig. 2).

Stellations of the icosahedron may all be derived from the cells enclosed within the twenty intersecting facial planes of the icosahedron. Besides the icosahedron itself, we will find 20, 30, 60, 20, 60, 120, 12, 30, 60, 60 cells of ten different shapes and sizes. The great icosahedron, discovered by Poinsot, is composed of all but

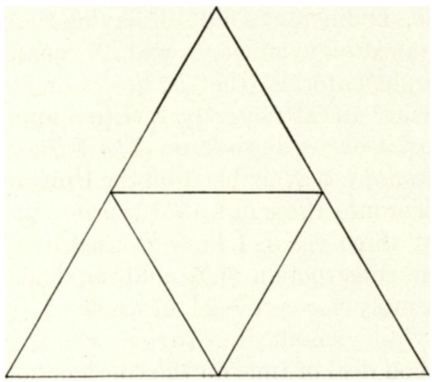

Figure 1

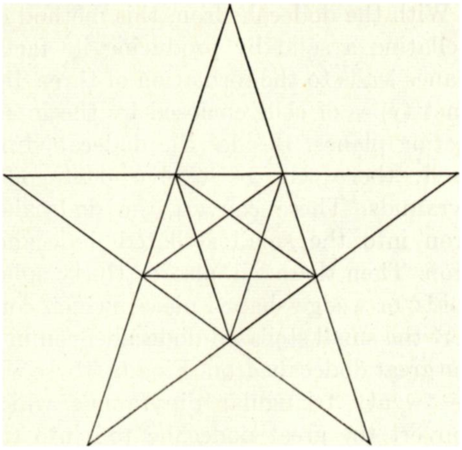

Figure 2

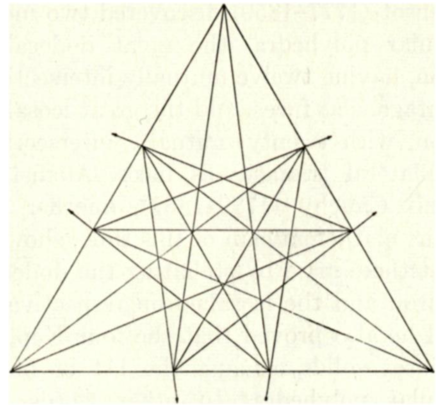

Figure 3

the last 60 pieces. Other stellations include a compound of five octahedra, a compound of five tetrahedra in two forms, *dextro* and *laevo*, and a compound of ten tetrahedra composed of both the latter forms combined. After these forms were discovered, the question naturally presented itself: How many stellated forms are possible? In 1900, Max Brückner, who published a classic work on polyhedra entitled *Vielecke und Vielfläche*, presented a number of new stellations of the icosahedron. Several more were due to A. H. Wheeler (1924). In 1938, H. S. M. Coxeter, in conjunction with P. DuVal, H. T. Flather, and J. F. Petrie, gave the question a systematic investigation. By applying a few restrictive rules suggested by J. C. P. Miller to determine what forms shall be considered properly significant and distinctive, Coxeter arrived at a total enumeration of fifty-nine, 32 different solids all having the full icosahedral symmetry, and 27 enantiomorphous forms (that is, *dextro* or *laevo*) having an attractively twisted appearance. Coxeter's work on *The Fifty-nine Icosahedra* is available from the University of Toronto Press in a 1951 reprint. In the past three years, I have worked out my own construction nets and methods of assembly for a model of each of these fifty-nine icosahedra. Anyone who spends a good deal of time on the construction of polyhedra will soon learn that no matter how many have been made, each one presents its own challenge and is a source of unique satisfaction upon completion.

The stellation pattern for the icosahedron is very interesting. It is most easily obtained by drawing one of the equilateral triangles that form the faces of the great icosahedron. On each side of this triangle we may locate two points dividing the sides of the triangle in golden ratio. Lines radiating from these points will give the pattern. (See Fig. 3.)

So far we have discussed the five Platonic solids and their stellations. There is another set of solids known as the Archimedean, or semiregular, solids. These all have regular polygons as faces but admit a variety of such polygons in one solid. There are thirteen of these and they are ascribed to Archimedes (287–212 B.C.), because he first enumerated them, though his work on them is lost. References to the work of Archimedes on this subject are found in the writings of Pappus, a mathematician of the fourth century A.D. Kepler was the first to formulate a complete theory concerning them.

It would be possible to stellate all the Archimedean solids by the same process as was indicated for the Platonic solids, namely, by producing each facial plane and imagining the stellated forms as being built from the various cells enclosed by the intersecting planes. But little investigation along these lines has been published.

The same may be said about still another set of polyhedra known as the Archimedean duals, which instead of being facially regular are vertically regular. Being duals, there are also thirteen of them. Two of the most interesting ones are the rhombic dodecahedron and the rhombic triacontahedron, both discovered by Kepler. Stellated forms of the first-named solid were studied by Dorman Luke and published in the 1961 edition of *Mathematical Models* by Cundy and Rollett. One of the stellations of the second-named solid is a compound of five cubes. Here, too, we find the golden ratio dividing the sides of each square face of the cubes. Some other stellations of the triacontahedron were investigated by J. D. Ede and published, without drawings other than the stellation pattern, in 1958 in the *Mathematical Gazette* in England.

Perhaps the reason for so little investigation of stellated Archimedean forms is, first, that here the work involved begins to assume the aspects of an endless quest; the possibilities become bewilderingly numerous. However, rather than seek to enumerate these seemingly infinite varieties of geometrical forms, we might ask a question of another kind. Is there, perhaps, a functional relationship between the number of faces of a regular or semi-regular solid and the number of distinct cells enclosed by the intersecting facial planes? Perhaps some day some mathematician may discover a relationship analogous to that of Euler's constant for convex polyhedra: $V-E+F=2$. Another reason for so little work on stellated Archimedeans is perhaps to be found in the fact that many of these are not particularly attractive or aesthetically pleasing. Mathematicians, after all, are human, and where investigation has no practical significance there must yet be some kind of motivation, such as beauty of form.

The study of polyhedra can be pursued from still another point of view, one which branches into the world of uniform polyhedra. A polyhedron is said to be uniform if its faces are regular polygons, which may be interlocking or intersecting, while all its vertices are alike. Again, it was H. S. M. Coxeter, working this time with M. S. Longuet-Higgins and J. C. P. Miller, who published a scholarly paper on the subject in 1954 entitled "Uniform Polyhedra" (*Phil. Trans. Roy. Soc. London*, A 264, 401–50). The five Platonic solids, the thirteen Archimedean solids, the four Kepler-Poinsot star polyhedra all belong to this set of uniform polyhedra. Such a polyhedron can be enclosed within a sphere, its center coinciding with the center of the sphere and all its vertices lying in the surface of the sphere. The planes of symmetry of the solid will thus partition the surface of the sphere into a tessellated network of spherical triangles. Those arising from the Platonic solids were first investigated by Augustus Ferdinand Möbius (1790–1868) in 1849. Hermann Amandus Schwarz (1843–1921) extended the theory to other tessellated networks of triangles on a spherical surface in 1873. W. A. Wythoff successfully exploited this theory to investigate polytopes, as they are called, in four-dimensional space, in 1918. The study of uniform polyhedra done by Coxeter and his associates was based on a systematic application of Wythoff's construction to all possible Schwarz triangles. According to Coxeter, "The earliest complete enumeration of convex uniform polyhedra was made by Kepler (1619), who observed that the definition includes also the prisms with square side faces and the antiprisms with equilateral triangular side faces." Thus the prisms and antiprisms belong to an infinite set since both have end faces which may be any regular n-gon. Some other historical notes supplied by Coxeter are: Two new uniform polyhedra were discovered by Edmund Hess (1843–1903) in 1878, 37 others by A. Badoureau in 1881, and 18 by Johann Pitsch in 1881, working independently. Brückner illustrated many of these in his book published in 1900. Between 1930 and 1932, Coxeter and Miller discovered 12 other uniform polyhedra. Between 1942 and 1944, Longuet-Higgins rediscovered many of these, including 2 not previously published. Thus Coxeter's enumeration published in 1954 lists 53 other uniform polyhedra besides the 9 regular and 13 semiregular, or Archimedean, solids a total of 75.

One remarkable new polyhedron is contained in Coxeter's 1954 list. It is exceptional, insofar as it is the only one which cannot be derived immediately from a spherical triangle by Wythoff's construction. It is the only known polyhedron that has more than six faces at each vertex. The vertices of two triangles, four squares, and two star polygons, a total of eight polygons, are found at each vertex of this strange polyhedron.

To quote Coxeter once more by way of conclusion, "We shall be much surprised if any new uniform polyhedron is found in the future." And again, "It is the authors' belief that the enumeration (namely 75 in all) is complete although a rigorous proof has still to be given." So the last word has not yet been written. And, as we find in every field of human inquiry, so here, too, mathematical investigation is still possible in the bewildering and beautiful world of polyhedra.

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The history of the dodecahedron

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Of the five regular solids, the dodecahedron has many claims to the most complex history. Unlike the others (cube, tetrahedron, octahedron, and icosahedron), which originated before historical records were kept, tradition assigns the discovery of the dodecahedron to Hippasus, a Pythagorean of the fifth century before Christ [1].* For his impiety in devising an addition to the perfect solids given by the gods he was lost at sea, according to the legend.

In the fourth century before Christ, when Plato (in the *Timaeus*) associated the other four regular solids with the four elements—earth, air, fire, and water—he assigned the sphere of the universe to the dodecahedron, possibly because there were no more elements to pair with another regular solid, but also because the volume of a regular dodecahedron inscribed in a sphere is closer to that of the sphere than the volume of any other inscribed regular solid.† In recognition of Plato's uses of them, incidentally, the five regular convex solids are often called the *Platonic bodies.*

* Numerals in brackets refer to the Notes at the end of the article.

† *Editorial Note.* The uninitiated will almost always intuitively but incorrectly believe that of a regular dodecahedron (a regular solid of 12 pentagonal faces) and a regular icosahedron (a regular solid of 20 triangular faces) inscribed in the same sphere, the icosahedron has the greater volume. Of the cube and regular octahedron inscribed in the same sphere, the cube has the larger volume.