CENTERS OF A TRIANGLE

(a set of construction activities for year 8 students)

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Problem

Two points A and B and a third point (G, resp. O, resp. H, resp. I) are given.
Construct the point C such as the third point is a center (center of gravity, resp. circumcenter, resp. orthocenter, resp. incenter) of triangle ABC.

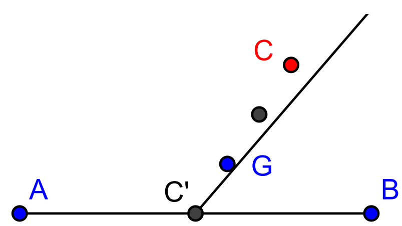
The aim of this activity is to learn to search a solution by analysis and synthesis:
- remind the theorem involved and illustrate it with a drawing,
- on the drawing search the properties susceptible to lead to a line of thought,
- write down the construction program and test it,
- discuss the conditions which the third point must verify to make the construction possible,
- how many possible solutions?

Test constructions will of course be useful.

1 Center of gravity

Easy construction from the midpoint C' of segment AB.
C can always be constructed: it belongs to line $C'G$ such as $GC = 2 \times GC'$.
We can test that G is also the thirds of segments AA' and BB' where A' and B' are the midpoints of AC and BC.

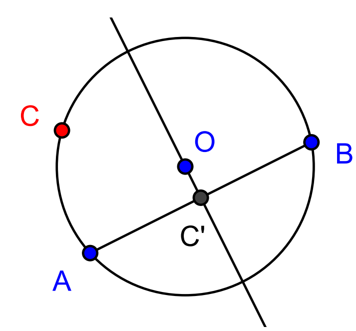
If G belongs to line AB then G does also; thus G must be chosen outside of line AB, and there is a unique solution (young students believe that this is always the case).



2 Circumcenter

Surprise! In general the problem has no solution (case often forgotten in our curriculum, but life provides us a lot of examples!).

If O is chosen on the perpendicular bisector of AB then there are an infinity of solutions: C can be any point on the circle with center O and going through A and B (of course A and B must be excluded).

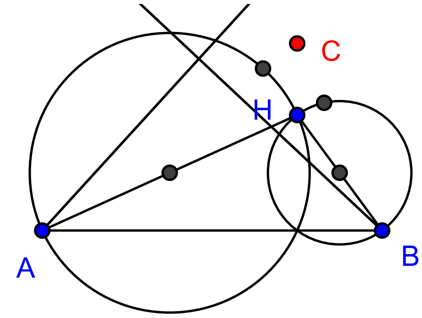


3 Orthocenter

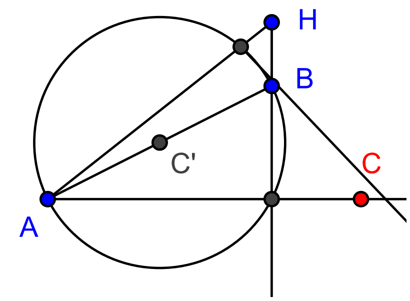
$ABCH$ is an orthocentric quadrangle, thus C is the orthocenter of triangle ABC.
Even if this result has been pointed out, this problem remains difficult for students who often "forget" that a right angle is inscribed in a half circle.

Thus there is always an unique solution, as long as H does not belong to the line AB.
It is not necessary to construct the perpendicular line to AB through H, but it is useful as verification.
If H belongs to the circle with diameter AB then $C = H$ (the triangle is rectangular).

first alternative
Construct the circles with diameters AH and BH; the intersections of lines BH and AH with these two circles are the feet of the two altitudes through B and A respectively, thus we can draw the two other sides of triangle ABC and we are done.



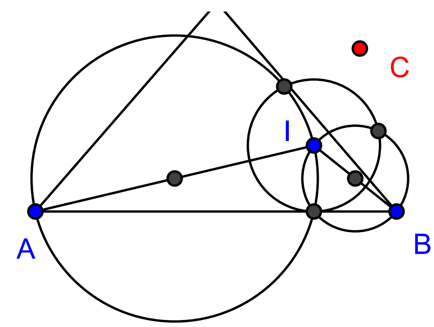
second alternative
We have only to construct the circle with diameter AB; the intersections of lines BH and AH with this circles are again the feet of the two altitudes through B and A respectively and we are done.



4 Incenter

If the construction is easy, this case remains the most difficult but also the most interesting.

To get the two other sides of triangle ABC we have to construct the symmetrical sectors of IAB and IBA wrt the lines IA and IB respectively.
The circles with diameters IA and IB intersect the incircle (tangent to AB) in two points which belong to the sides and we are done.



Now how to choose I such as the problem has a solution (which is then unique if I is not on line AB)? Quickly the students discover that if C is chosen *to far* from line AB, then C jumps on the other side of AB and the I becomes an excenter. Likewise they quickly restrict the position of I inside the strip with sides the lines perpendicular to AB through A and B.

To find the domain where I must be chosen is it interesting to prove the following result:

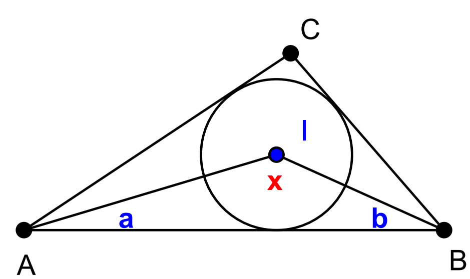
Property: The three sectors with vertex the incenter defined by the three interior bisectors of a triangle have obtuse angles.

Proof: (using the sum of the angles of a triangle)

$$x + \frac{a}{2} + \frac{b}{2} = 180^\circ$$

$$x = 180^\circ - \frac{a+b}{2} > 90^\circ$$

because $$\frac{a+b}{2} < \frac{a+b+c}{2} = \frac{180^\circ}{2} = 90^\circ$$



It is now clear that I must be inside the circle with diameter AB. This result is not obvious at all!