How "Shaky" Is The Jessen's Orthogonal Icosahedron ?

("shaky" = infinitesimally movable)

Jessen's Orthogonal Icosahedron

"A shaky polyhedron constructed by replacing six pairs of adjacent triangles in an icosahedron (whose edges form a skew quadrilateral)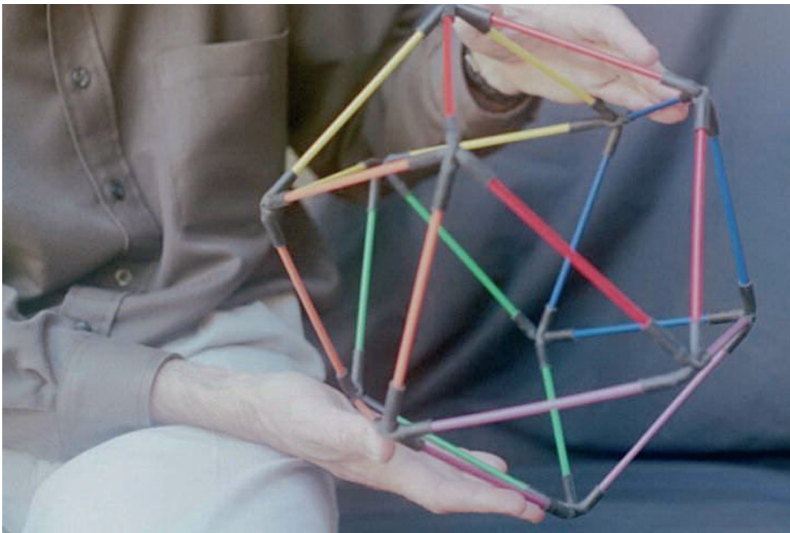 with pairs of isosceles triangles sharing a common base. The polyhedron can be constructed by dividing the sides of the icosahedron in the golden ratio (as used in the construction of the icosahedron along the edges of the octahedron), but by reversing the long and short segments".

You will find this upper statement on the webpage:

http://mathworld.wolfram.com/JessensOrthogonalIcosahedron.html

Apart from the fact, that the construction-description is not correct, (see the correct construction on the left below) we are also faced with the question, if you would intensively study the solid body, if it can be shaky at all?

The length of the radius of the circumcicle of the equilateral triangle (a) is the same length as the half of the short diagonal of the rhombus.

1

"Vector equilibrium"

„The vector equilibrium (VE) is an omnidirectional equilibrium of forces in which the magnitude of its explosive potentials is exactly matched by the strength of its external cohering bonds" (430.03)

"It is a hypothesis of synergetics that forces in both macrocosmic and microcosmic structures interact in the same way, moving toward the most economic equilibrium packings. By embracing all the energetic phenomena of total physical experience, synergetics provides for a single coherent system of geometric principles". ...(209.00)

„The vector equilibrium is a condition in which nature never allows herself to tarry. The vector equilibrium itself is never found exactly symmetrical in nature's crystallography. Ever pulsive and impulsive, nature never pauses her cycling at equilibrium : she refuses to get caught irrecover- ably at the zero phase of energy. She always closes her transformative cycles at the maximum positive or negative asymmetry stages."

R.Buckminster-Fuller "Synergetics".

The VE, a torsion polyhedron, which has been brought to light by Buckminster-Fuller (Jitterbug) in the 1940's, offers several marvelous revelations. Besides it ´s ability to transform from an octahedron to a cuboctahedron and vice versa, it becomes apparent, that the Jessen´s orthogonal icosahedron, is exactly the middel position of, the VE on its way between the octahedron to the cuboctahedron (Pict. 1, red).

Pict.:1

The icosahedron

The Jessen´s orthogonal icosahedron

The golden icosahedron

Pict. 2

Pict. 2a

Pict. 3
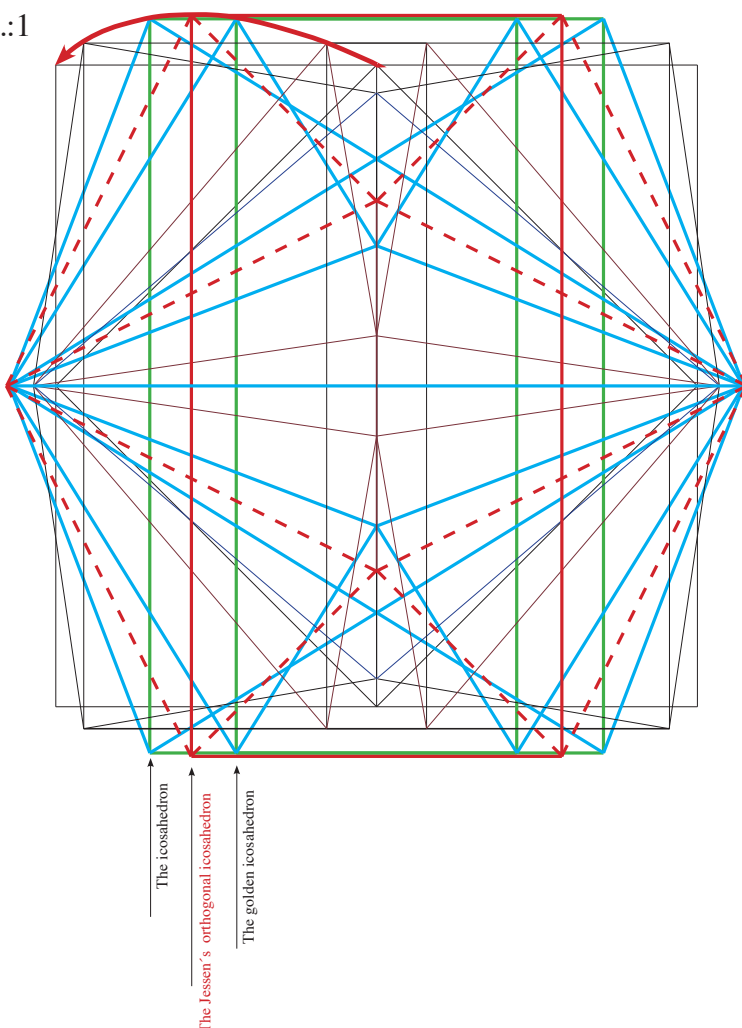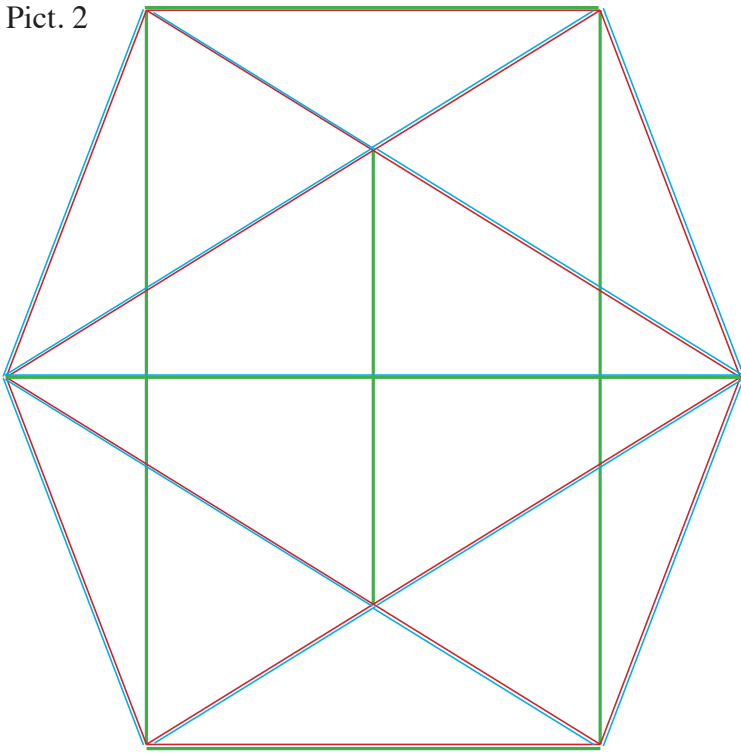
As we have a look at this moving-path, we see, that there are many more pairs of concave icosahedra (Pict.: 1, blue).

The "cover form" of the first is the regular icosahedron. In Pict. 2. you can see the convex and the concave solid (red and blue) as well as the 3 space planes of the inner 5 space-cross of the icosahedron (green).

Now let's have a look at the behavior of these three "special space plain's". The amazing fact is that this concave icosahedron can appear in a second space form, (Pict. 3) keeping all the triangular plains the same (8 equilateral triangles, 12 isosceles golden triangles), (Pict. 2a).
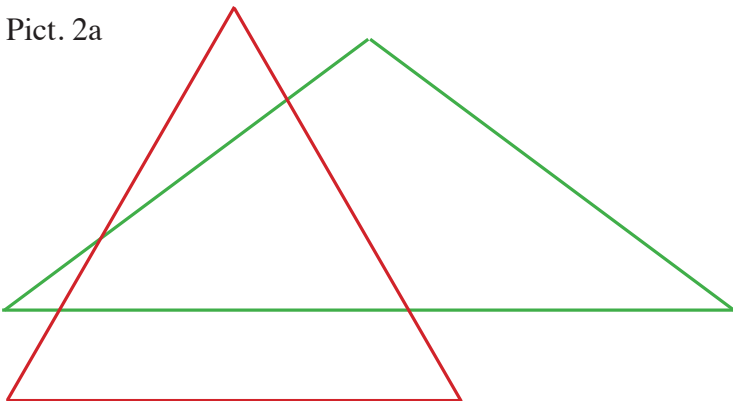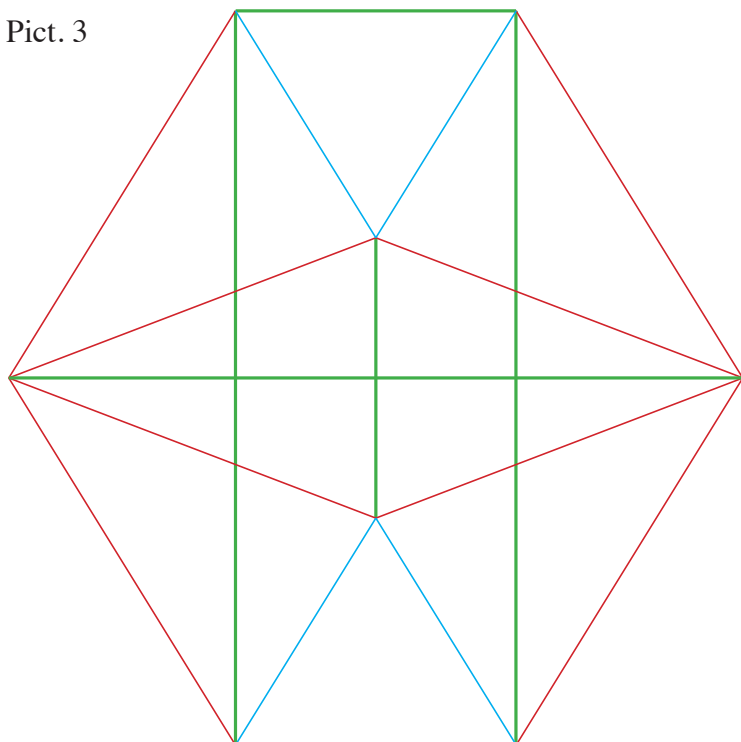
The volume of the surface remains unchanged, but the space-volume of the solid has changed.

The "cover form" in position 1 is, like we said, a regular icosahedron. Now – in position 2 (see below) it has transformed into a golden icosahedron.
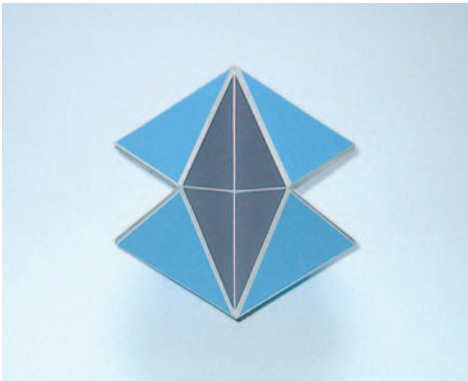
A golden icosahedron consists out of 8 equilateral triangles and 12 isosceles golden triangles, which are orthogonally opposite.

Golden icosahedron

3
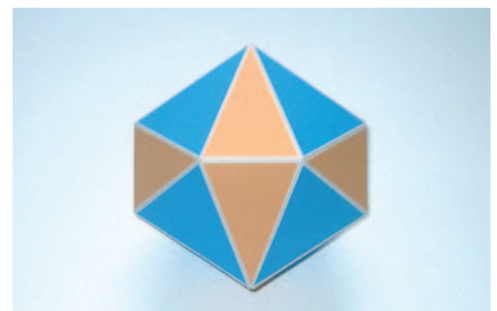
concave-solid
of the icosahedron

Isn't it amazing, that a solid, a polyhedron, whose faces are steady in their dimensions and in the position between each other, can be found in two different space forms?!

How and what is the path from the first to the second position?

concave-solid
of the golden
icosahedron

Pict.: 4

The icosahedron

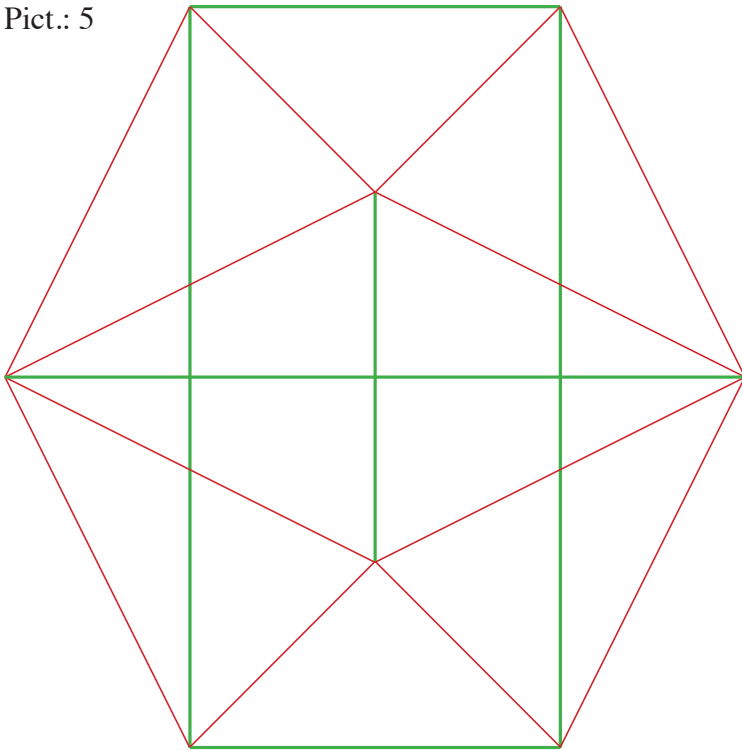The medium twenty surface solid

The golden icosahedron
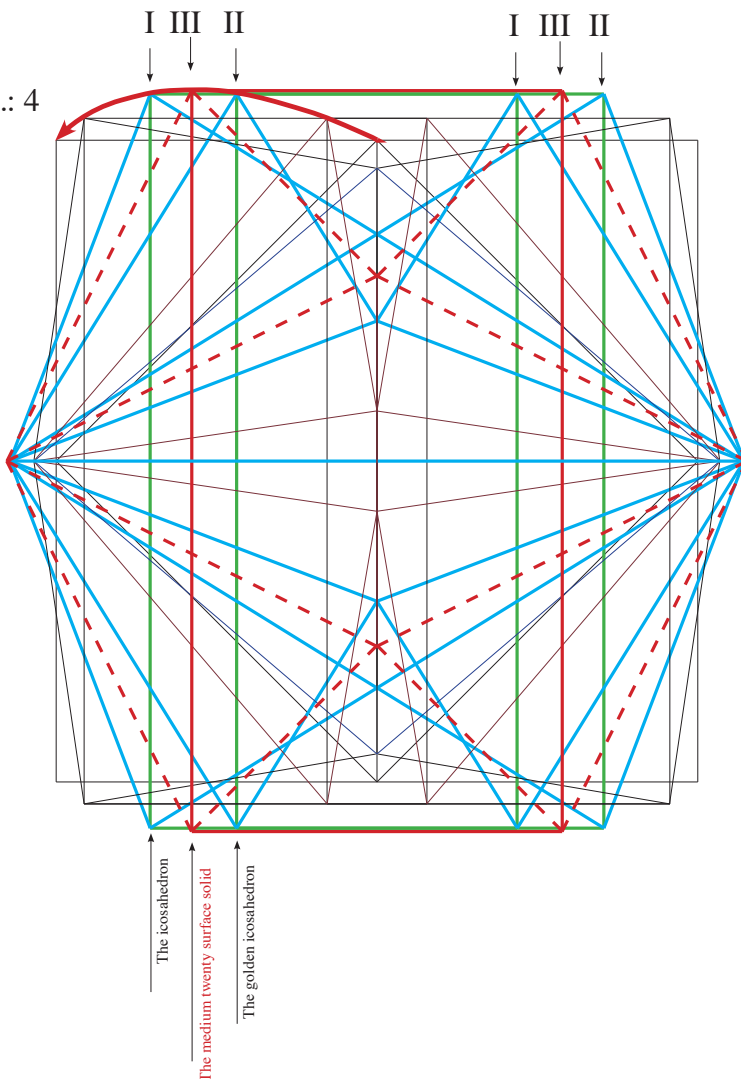
The middel position

Icosahedron and golden icosahedron are one of the many „connected-solid-pairs" of the VE on it's path from octahedron to cuboctahedron.

You can see that the longer sides (left I and II and right I and II - Pict.4) of the space plains (green) stay the same (in both the icosahedron and the golden icosahedron) while the distance of the short sides (top and bottom) decrease in the golden ratio.

4

Pict.: 5

As we have a look at the middel posi-
ton of the VE (Pict.:4, red) we can see
something very important!

To get from the one (Pict. 2) to the other
position (Pict. 3) there is another conca-
ve icosahedron, holding it's own indivi-
dual space plain.

This is the Jessen´s orthogonal icosahe-
dron. I call the "cover form"of this solid
the medium "twenty surface solid"
(Pict. 5)

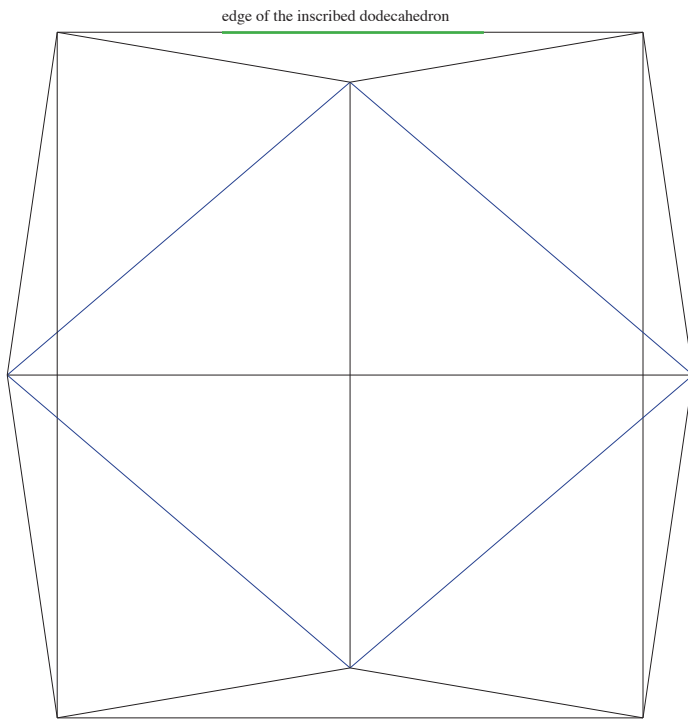The JOI is a concave "brother-solid" of
the medium "twenty surface solid"

By this individual space plain (Pict. 4,
III, red), the longer sides (left and right)
are extended minimally in their length.

Something else comes about:

This individual space plain has trans-
formed to a double square, whose short
edge is at the ratio of approx 1:sqrt 1,5
to the octahedron's edge 1. I call this the
"medium icosahedron" (Pict. 5)

concave-solid
of the medium
"twenty surface solid"

As a result of this, we can say:
"Yes" there are a lot of pairs of concave
icosahedra which change their space-
form in two positions, holding their sur-
face-volume whilst transforming their
space- volume. But it is impossible that
the way from one position to the other,
can be made in one movement (there is
no other movement possible than the VE
movement) unless the longer concave si-
des change in length (Pict.4, III), and/or
the plains bend.

Pict.: 6

edge of the inscribed dodecahedron

the big icosahedron

Pict.: 7

the small icosahedron

long edge

dodecahedron edge

major   minor

You can find another „connected-solid-pair". I call it the "big" and the "small icosahedron" (Pict. 6 + 7).

The big icosahedron is very interesting. The space plane dimensions in our special space-cross emerge from the proportions of the inscribed dodecahedron.

The short edge (green) of the space plane of the small icosahedron is equal to the minor of the short edge of the dodecahedron that is inscribed into the big icosahedron (See drawings on page 6 to 10).
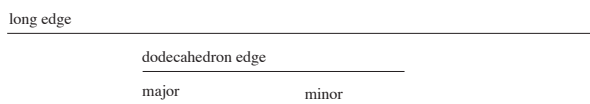
Again we have two different polyhedra affecting their space forms with constant volumes.

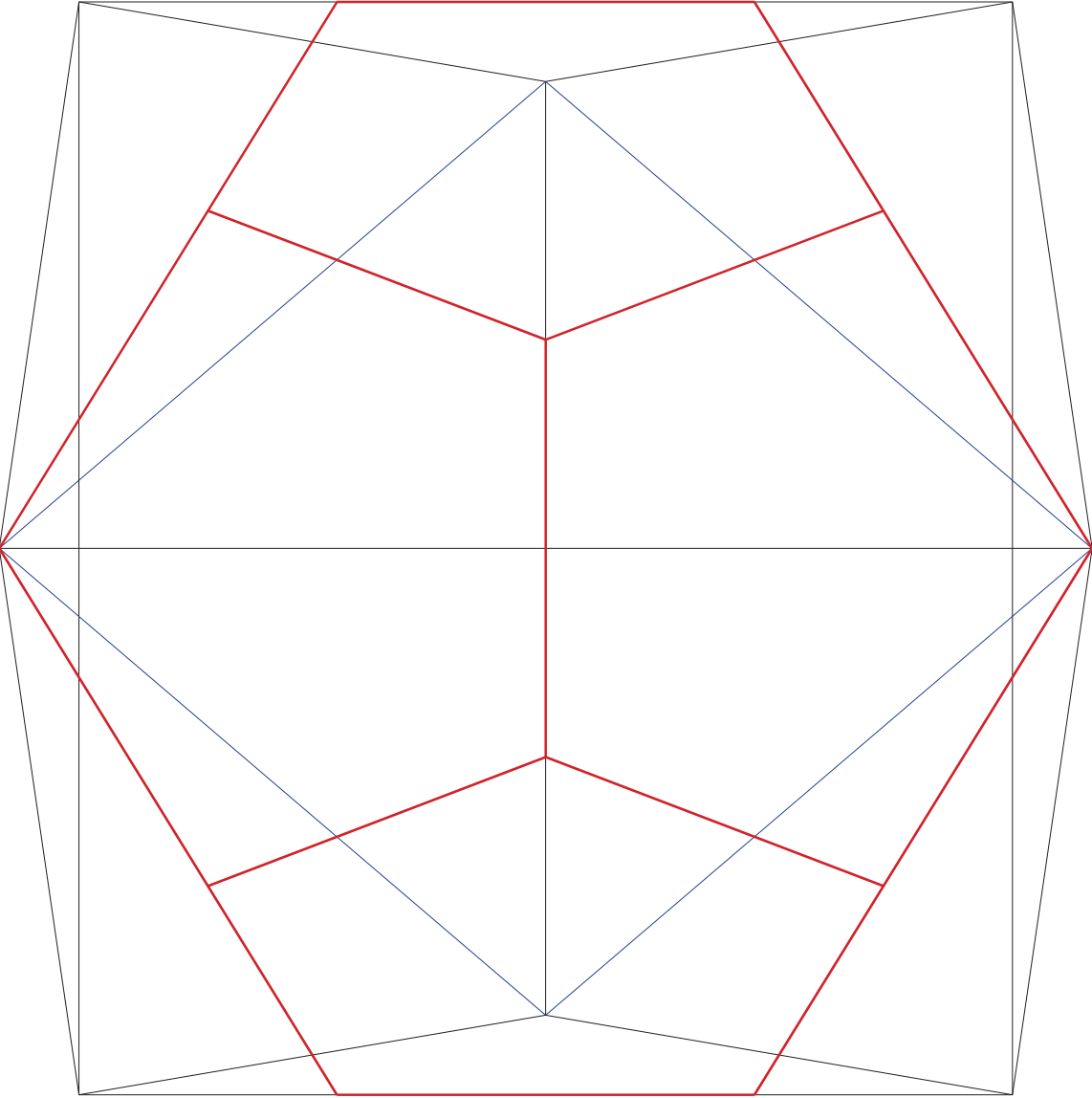The outer form changes, but the inner volume stays the same.
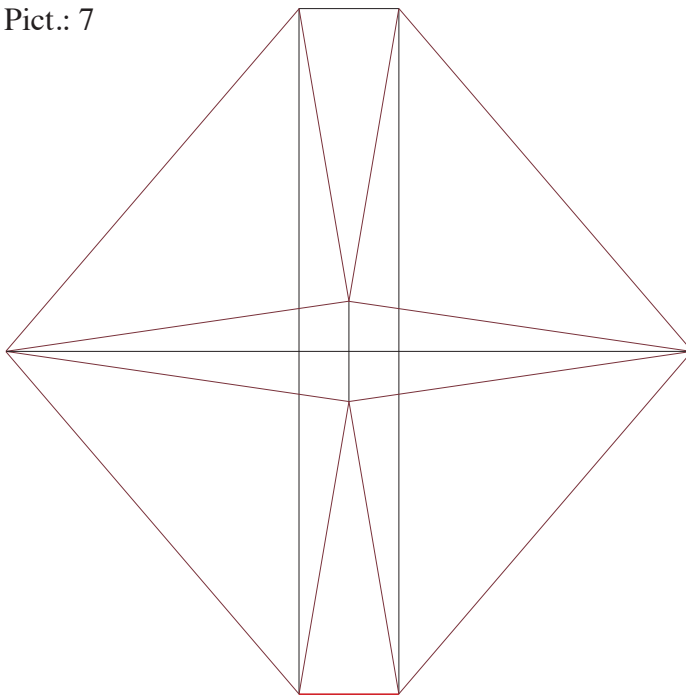
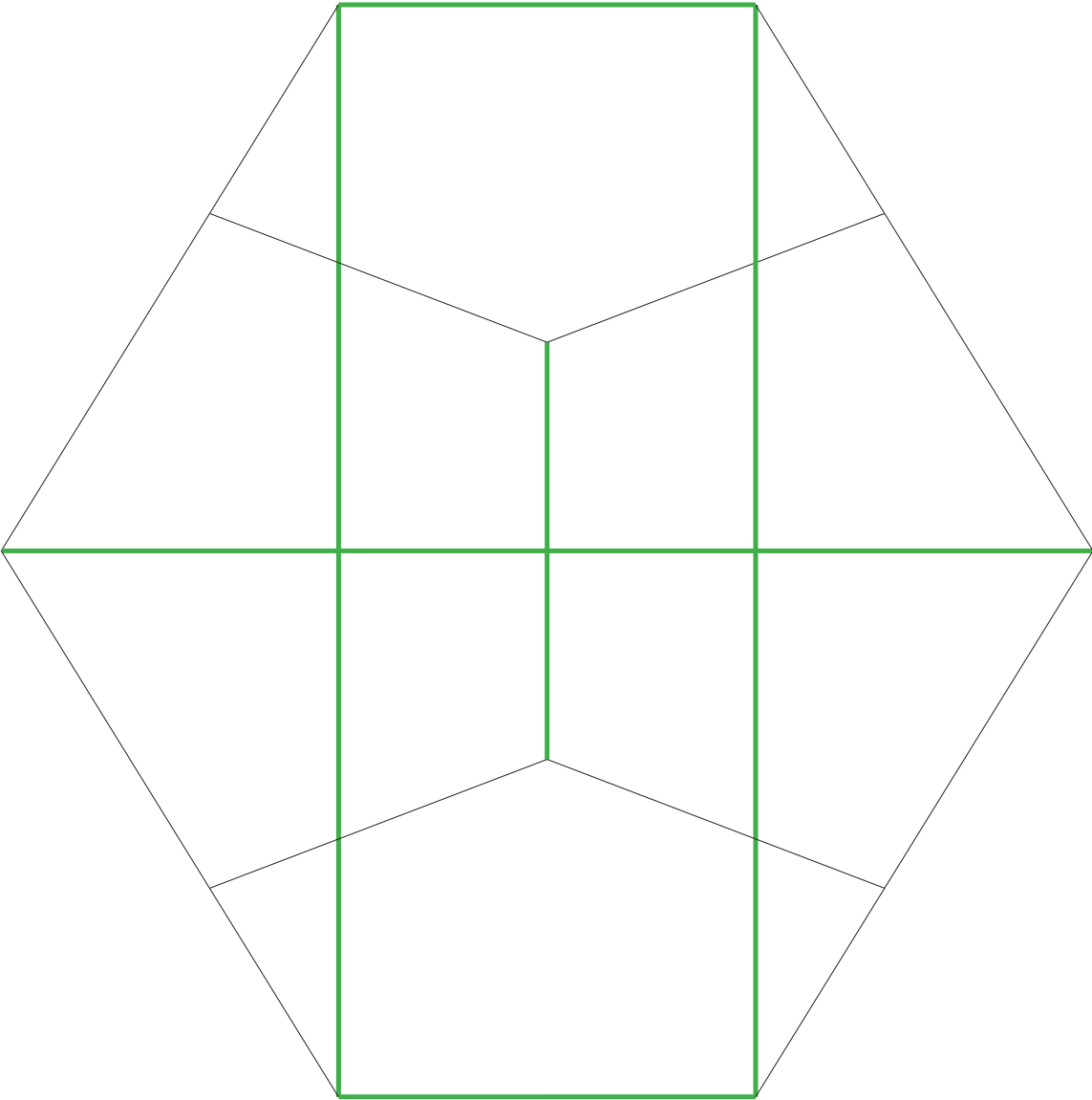
concave-solid
of the big icosahedron

concave-solid
of the small icosahedron

The „big icosahedron" and the
inscribed dodecahedron

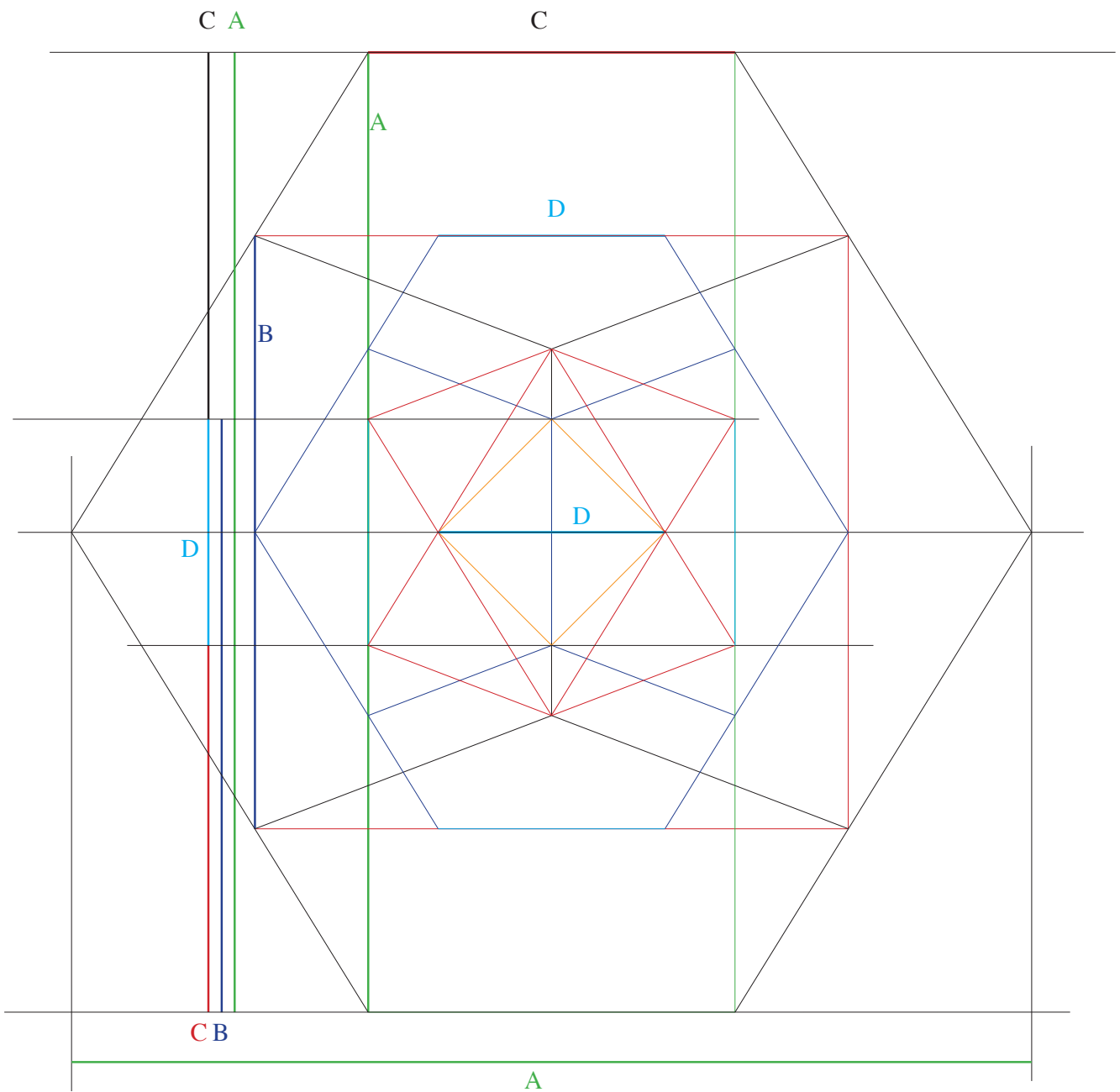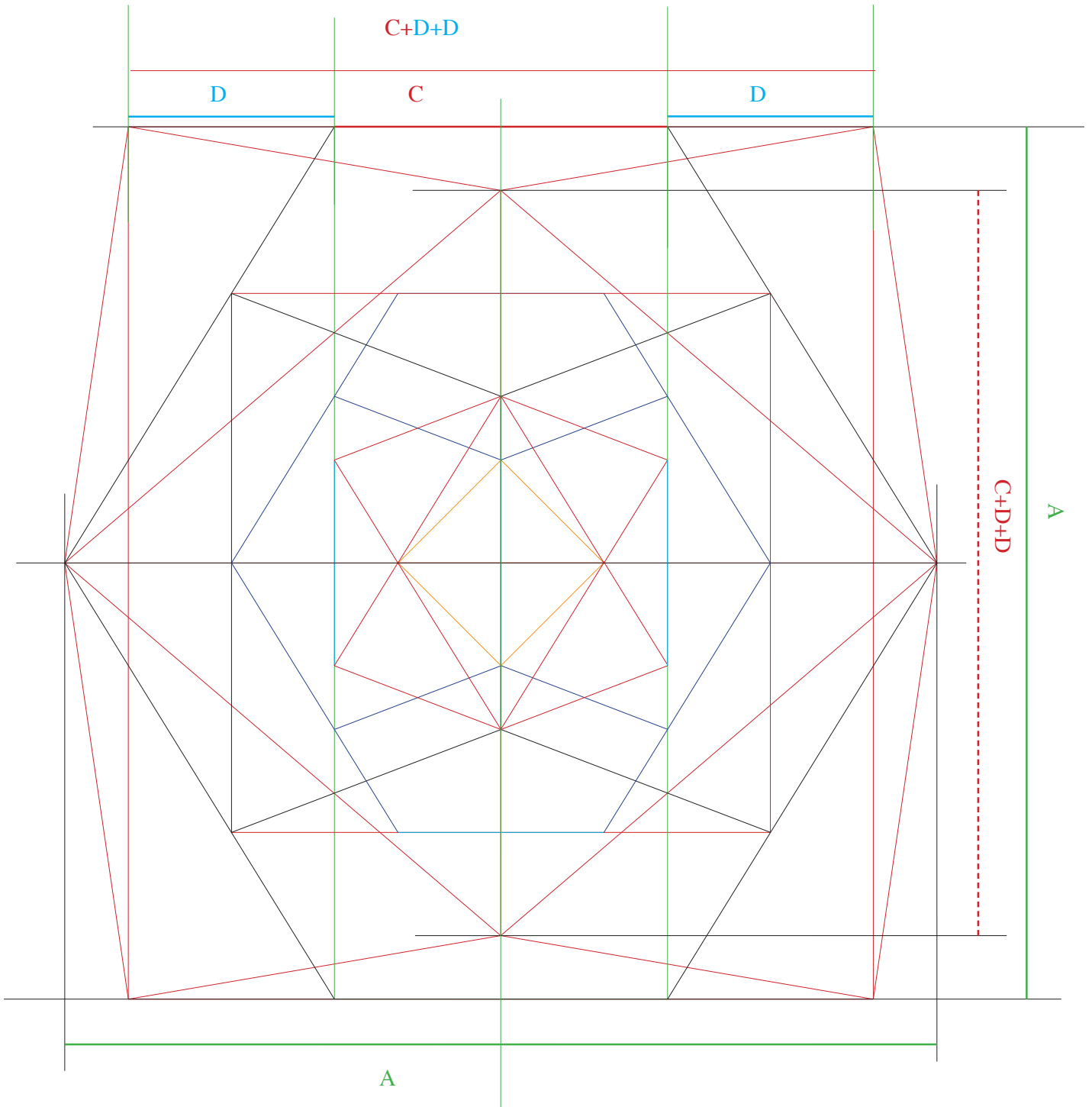The space planes inside the dodecahedron

The lengths of the rectangle
of the dodecahedron space planes

minor of the total distance

minor

major

the new total distance

the total distance

minor

minor

A = the total distance of the space plane
B = the major (coextensive edge of the polar cube, also marking point of the vertex of the inside icosahedron)
C = minor of the long distance of the space plane rectangle, (coextensive outside edge of the dodecahedron,
    also marking point of the vertex of the inside icosahedron)
D = minor von B (coextensive edge of icosahedron and edge of inside dodecahedron)

With the assistance of the *"Goldenen sections"* A, C and D we can construct the big twenty surface solid.
The measured distance of the six long edges are hidden in the dodecahedroen space planes.

The movement of the "vector equilibrium" from octahedron to cubeoctahedron. Length of the edge of the octahedron = 1.

Seven possible positions - seven different polyhedra. Three complementary pairs - one single.

The cubeoctahedron

The big twenty surface solid

The icosahedron

The medium twenty surface solid

The golden icosahedron

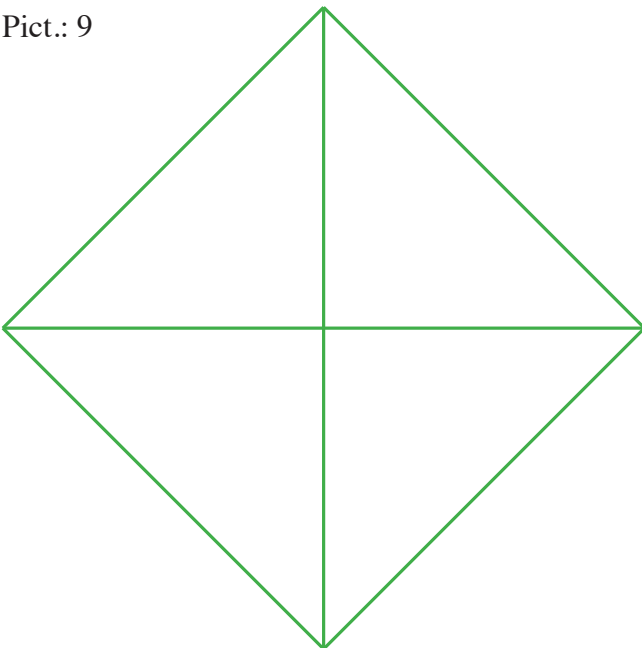The small twenty surface solid
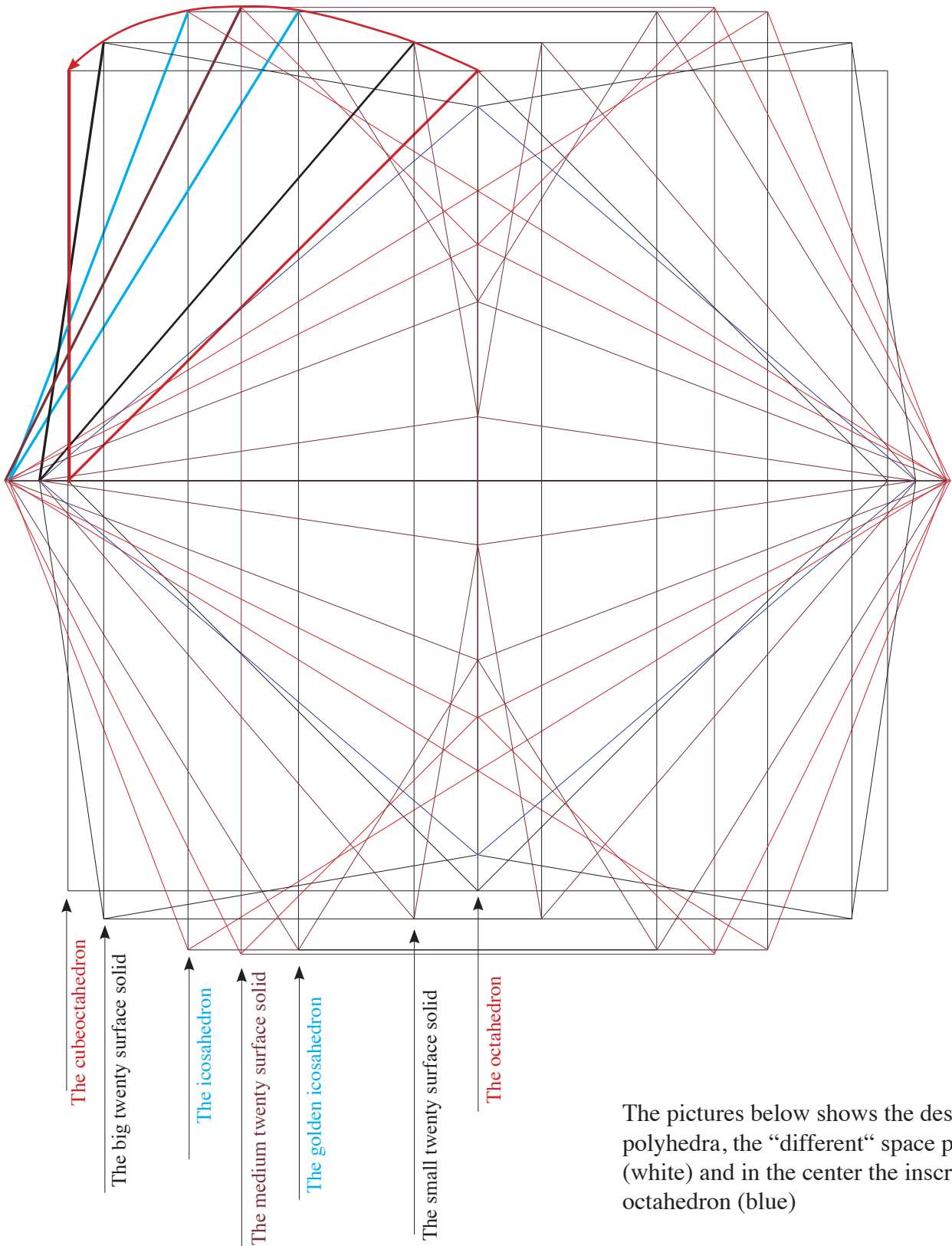
The octahedron

The pictures below shows the described polyhedra, the "different" space planes (white) and in the center the inscribed octahedron (blue)
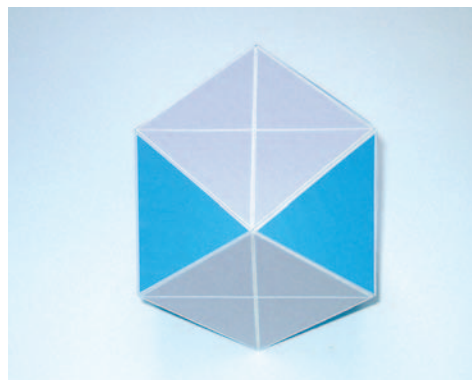

12

Pict.: 8

The last „connected-solid-pair" is made by cuboctahedron and octahedron (Pict. 8 + 9)

The 3 space planes of the cuboctahedron are squares with an edge-length 1*sqrt2. The 3 space planes in the octahedron have disappeared in the diagonals and the 6 corners of the octahedron.
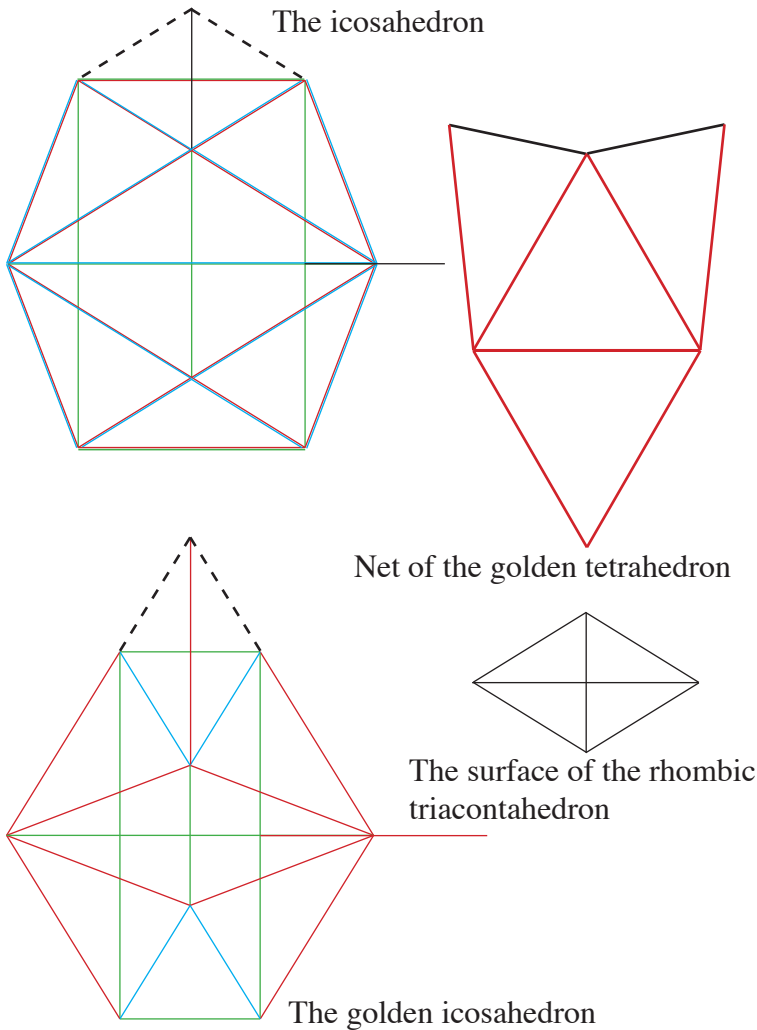
Conclusion

The moving path of the VE can be divided in pairs of icosahedra which belong together as „connected-solid-pairs" (except the octahedron, the cuboctahedron and the individual medium icosahedron) – although you can define octahedron and cuboctahedron as transformed icosahedra.
I will come back to this later.

Two significant „connected-solid-pairs" have come to light. But there are as many pairs as the length of the movement-path allows.

Pict.: 9

We know that the octahedron-cuboctahedron-pair can fill the space without gaps. What about the other pairs? And what about the medium icosahedron? How do they fit in the VE-family?

By cubeoctahedron and octahedron concave and convex-solid become one.

The icosahedron

Net of the golden tetrahedron

The surface of the rhombic triacontahedron

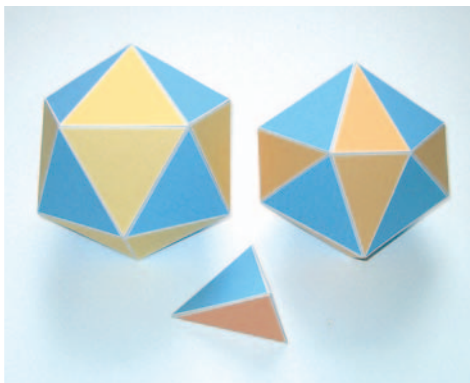The golden icosahedron

The Space-Filling Icosahedra:

Icosahedron And Golden Icosahedron

If we add a golden tetrahedron*
to our pair of icosahedron and golden icosahedron, we can fill the space.

2 of its 4 faces are golden isosceles triangles, where the short edge relate to the long edge in the golden ratio.

The dimensions of the tetrahedron's edges are a combination of both short edges of the space planes of both solids with the basic-length 1.

If you would combine both short edges of the icosahedron and the golden icosahedron, as diagonales, you can build a golden rhombus. This rhombus is one of the surfaces of the rhombic triacontahedron which belongs to these both solids.

Icosahedron, golden icosahedron and golden tetrahedron.

The golden icosahedron

The golden tetrahedron
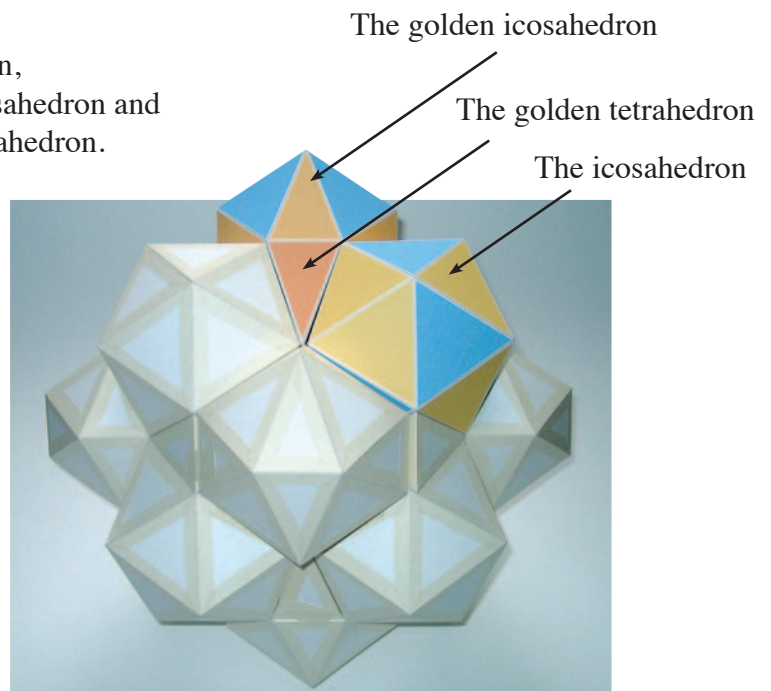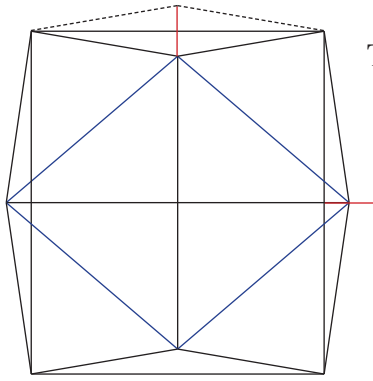
The icosahedron

The big twenty surface solid

Net of the tetrahedron

The small
twenty surface solid

The Big
And The Small
Icosahedron

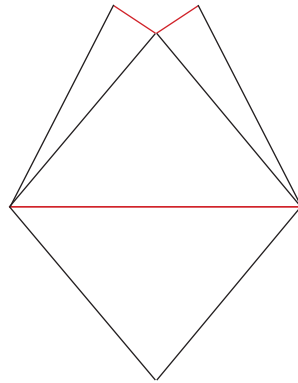If we add a fitting tetrahedron to the big and the small icosahedron we can fill the space too.

The dimensions of the tetrahedronsides arise out of the combination of both small sides of the relating space edges of both solids with the basic length 1, just as we described before.
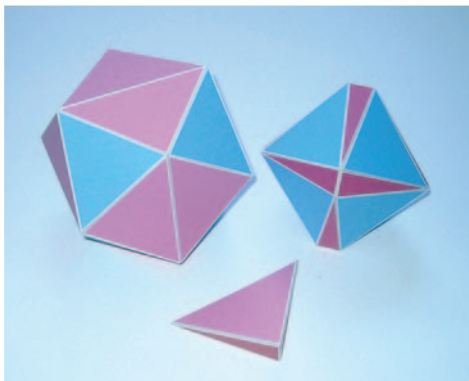
Big twenty surface solid, small twenty surface solid and the corresponding tetrahedron.
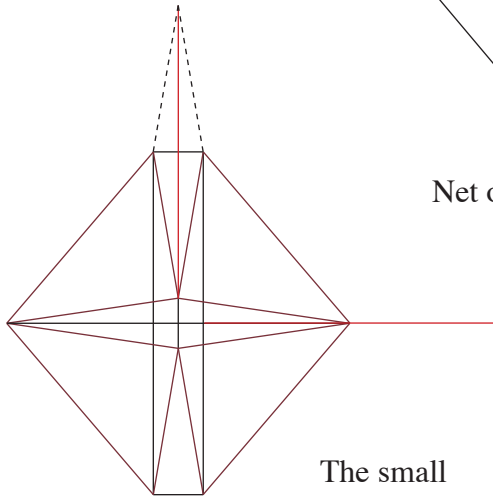

The small
twenty surface solid

The tetrahedron

The big twenty
surface solid
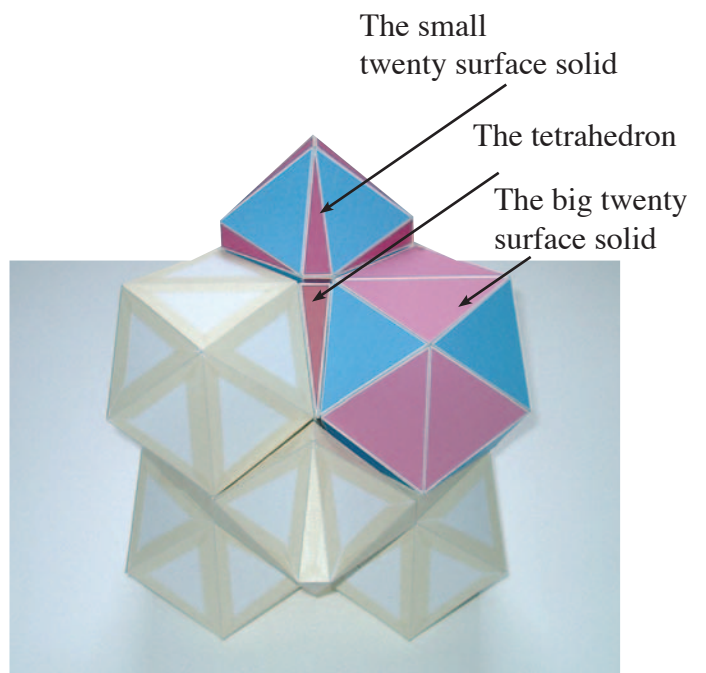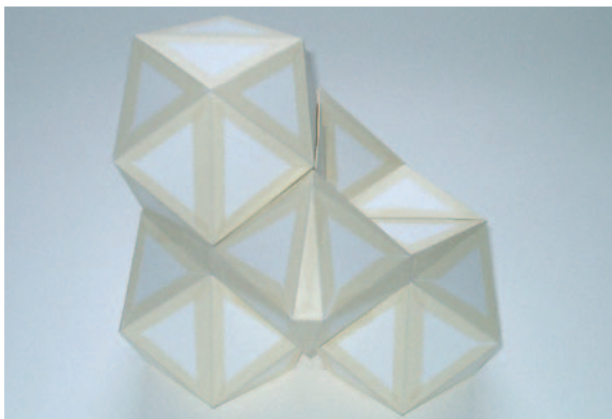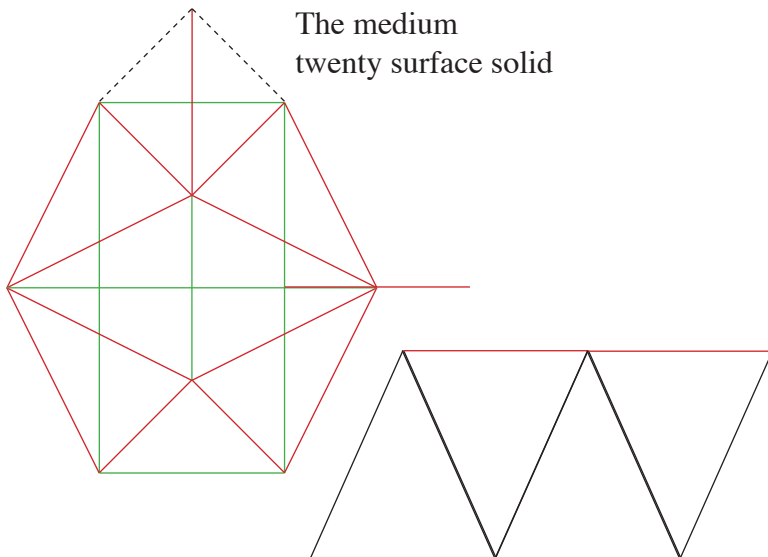
The medium
twenty surface solid

Net of the tetrahedron

Pict.: 10
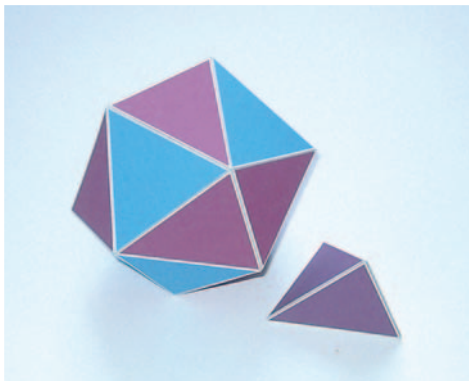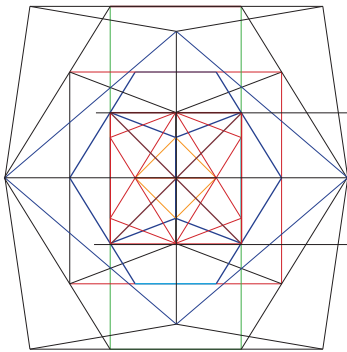
The medium
twenty surface
solid and the
corresponding
tetrahedron.

The Medium Icosahedron

To fill the space, not leaving any gaps, the medium icosahedron needs only one tetrahedron. The sides of the tetrahedron are isosceles triangles. The shorter edge of each triangle has to be at a ratio of approx. 1:sqrt1,5.

A square emerges out of the diagonal-combination of both short sides of the space plane.

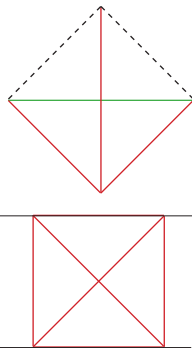The square-edge-length is the same as the short edge length of the hidden dodecahedron within the big twenty surface solid. The whole square face is one of the 6 faces of the cube that is inscribed to the dodecahedron (Pict. 10).

Combining these two irregular solids (icosahedron and tetrahedron) we CAN fill the space without leaving any gaps. If we do it with regular solids (icosahedron and tetrahedron) it won´t work!

And here we must realize that this „medium twenty surface solid" is the convex "brother solid" of the Jessen´s orthogonal icosahedron!
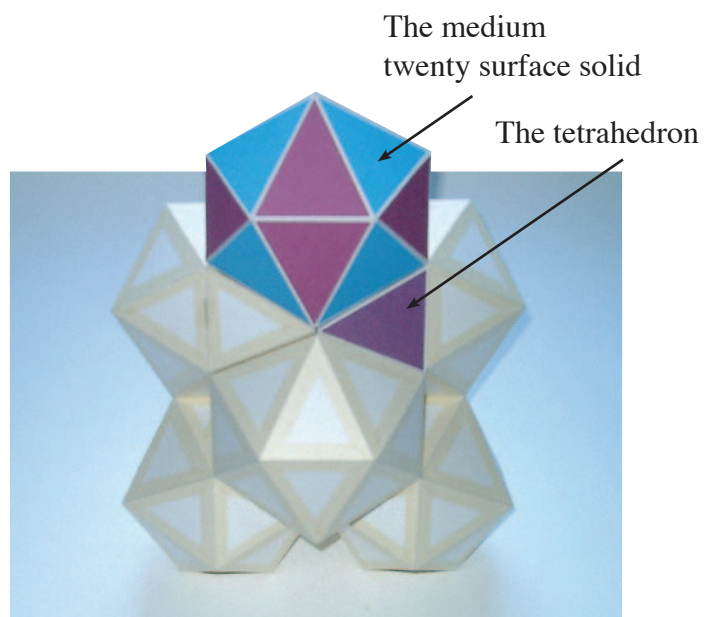
The medium
twenty surface solid

The tetrahedron

The octahedron
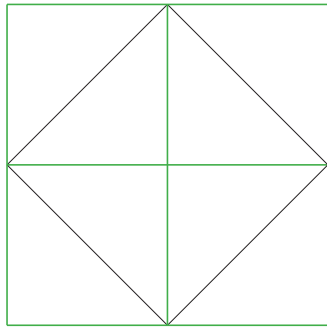
The cubeoctahedron

Pict.: 11

Pict.: 12
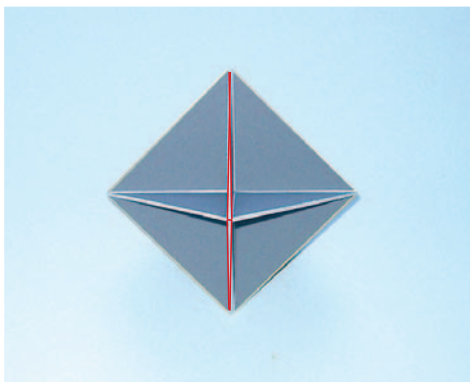
Pict.: 13
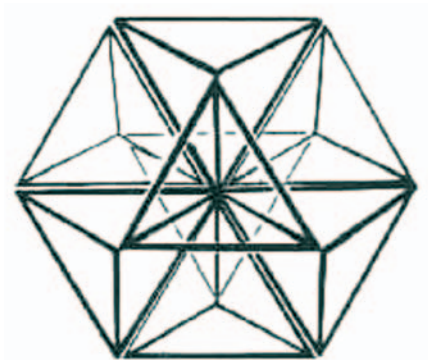
Octahedron
And Cuboctahedron

We mentioned earlier that octahedron and cuboctahedron can be defined as transformed icosahedra.

You can also imagine the 6 square faces of the cuboctahedron as diagonal connections (1*sqrt2) of two rectangular triangles. By doing so, the cubeoctahedron wil consist out of 20 faces (Pict. 11).

The 12 edges of the octahedron build a line of the 12 variable triangular faces of the VE and so, you get 20 faces as well (Pict. 12).

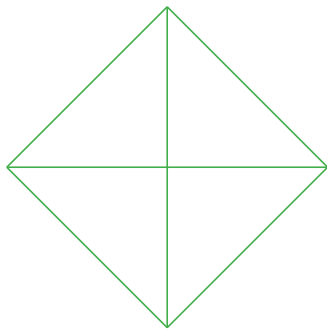You may also consider the cuboctahedron being built out of 8 regular tetrahedra and 6 half-octahedra (Pict. 13).

This shows you that all edges and connecting sections have the same length to the centre of the solid, namely the basic length 1.

Conclusion

By what we just have seen, we can conclude, that the VECTOR EQUILIBRIUM is worth it´s name in all it´s splendor!

It is an balancing instrument, similar to the scales. The forms of the weight objects, left and right, can be different, (space-volume) eventhough they have their weight in common (surface-volume). They are pairs which belong together. With help from a third element: the belonging tetrahedron, they can fill the space without leaving any gaps.

By seeing these facts, again we are faced with the orderly wonders of space-geometry.

© D. Junker / September 2008

17

Some Thoughts about the movable polyhedra in correlation to the „bellows conjecture"

Abstract. „We show that any continuous flex that preserves the edge lengths of a closed triangulated surface of any genus in three-space must flex in such a way that the volume it bounds stays constant during the flex." (The bellows conjecture, Connelly, Sabitov, Walz, 1997)

1.

There are concave polyhedra which can be made and/or already exist; they can come about in two or more space-forms. Eventhough they keep there SURFACE-volume, they change in SPACE-volume.

The expectation that all these polyhedra are flexible – somehow movable –, is not applicable in every case.

It is important to focus attention to the path on which the movement takes place; from one space form (with the biggest space volume in position 1) to the other space form (with the smallest space volume in position 2). Possibly, in some cases there are moving-paths in between the previous positions described. (Refer to article "How Shaky Is The Jessen´s orthogonal icosahederon?")

1.1.

Concerning this moving-path:
two categories of polyhedra exist.

1.1.1.

The first category is called "shaky", "infinitesimally flexible or movable" but "multi-stable" at the same time.
Whilst passing the moving-path there are many stable forms possible, such as: Position 1 (biggest space volume), position 2 (the smallest space volume).

The path from one stable form to the other can only be run through, when several hinge-edges change minimally, that means the surface volume and the length of the hinges of the matching solids can vary.

The Jessen´s orthogonal icosahederon SEEMS to belong to this category, but because of its dihedral angles it is not shaky at all and can only appear in one form!

The "Siamese Dipyramides" of Goldberg really belong to this category.

In other words: These polyhedra are not flexible in the sense of geometry. They have the special nature to manifest in different space forms whilst keeping their surface volume and changing their space volume.

The terms "Shaky" or "infinitesimally movable" are misleading. Because in this category of solids the movement cannot happen without changing the surface of the solid.

In other words; if you would make such a solid out of strong, steel plates (very thin) of which the surface stays fixed (coplanar) and the hinges cannot stretch, either sideways or in length, but only using their function, then you can conclude that NO polyherdron of this category would be possible to move at all.

1.1.2.

The second category represents a "continuously movable linkage". The moving-path from position 1 to position 2 can be passed in one movement, without changing hinge-edges or surfaces.

Position 1 (biggest volume) can be moved to position 2 (smallest volume) and vice versa. Both positions are stable and any position within the moving-path as well.

Belonging to this category: "Shaking And Toppling Octahedron" of Wunderlich. A fact in these catagories is that the surface volume stays the same whilst the solid volumes changes.

1.1.3.
If such a solid is really – hermetically – closed and airtight, then such a solid can not move.

When "closed triangulated surface" in the "bellows conjunction" means airtightly closed, then they would not be able to move at all.

1.1.4.
If "closed" means "constantly coherent at its edges and also not changing its surface-volume", then the soldis would be able to move, but only if the air would be able to get in-and-out.
How does the air get in and out?

1.1.5.
Here is the following applicable:
"It is also true that any continuous flex that preserves the edge lengths of a coherent triangulated surface of any genus in three-space can flex in such a way that the volume it bounds change during the flex."

1.1.6.
Connelly, Sabitov and Walz described in the "Bellows Conjecture" in 1997:
"All flexible polyhedra keep a constant volume in their movement."
All of our studied examples, so far, have shown the contrary.

2.
Another category is the continuously movable linkage, where the space volume really is constant all the time.
The "Connelly Sphere" and the "Steffen's Polyhedron" are the only known flexible polyhedra of this category.

Either these two polyhedra are the only flexible polyhedra – then the JOI is NOT shaky – or the bellows conjecture is not "generally applicable" but only applicable to certain categories of polyhedra.

The "Bricard's Octahedra" shall not be mentioned here, because they are polyhedra whose outer faces are not closed. Some of the surfaces penetrate others on the path of movement.

A further exception in terms of flatness is the VE (Vector equilibrium) of Buckminster-Fuller. Here the corners (points) of the faces work as hinges, not the edges (lines).

3.
The last category is formed by all kaleidocycles if they have a closed polyhedric form in their locked position. This polyhedric form can be convex or concave. But consider that polyhedra of this category are dismantled to fraction-polyhedra, which are conjugated on several edges to build a united link chain.
The "Schatz's Cube Belt" is the most famous solid of this category.

Conclusion:

1. The term "movability" applied to certain categories of concave polyhedra is not completly applicable with the "bellows conjecture".

2. The term "closed" in the sense of "hermetically closed" refers rather to physics, where the "real volume" (maybe a bellows) is meant – not an imaginary volume in the sense of geometry.
How does the bellows conjecture act in empty space without any air?

3. The terms "shaky" and "infinitesimal-ly" are misguiding. All really flexible solids are always just infinitesimally movable. The solids which are declared to be infinitesimally moveable within circles of Math and Geometry are not more than "multi-stable".

That means, they can manifest in va-rious space forms, but on their moving-path between two positions they have to change their surface-volume or they must give up their coherency.

4. In geometrically terms there are indeed flexible polyhedra, which change their volumes while moving, but keep coherent and maintain their surface-volume.

D.Junker, Okt. 2008