REVIEWS

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"Invertible" Polyhedron Models. Distributed by Snyder Engineering, 7552 Dumas Drive, Cupertino, California 95014.

Reviewed by **Gerald L. Alexanderson and Jean Pedersen**

We recently discovered the existence of some polyhedral models originating in what was once East Germany and we would like to call them to the attention of enthusiasts of polyhedral geometry who may not have discovered them yet. The seven models are large—suitable for demonstrations to groups of students and mathematicians—colorful and meticulously constructed with cloth hinges at the edges. Moreover, some have internal magnets allowing for easy transformation from one state to another. The models, all Platonic solids in their original state, are each decomposable into pieces that can be reassembled to form "stellations" having the same underlying symmetry as the original polyhedron. The process of transforming them is a kinetic process with interesting intermediate states. And, in almost any state, they are stunningly beautiful.

There is no standard language of which we are aware to describe the kinds of dissections involved, though there is a literature on these polyhedra, albeit, it seems, only in German. One such booklet is *Umstülpmodelle der Platonische Körper*, by Wolfgang Maas and Immo Sykora, published by Kaspar Hauser Therapeutikum, Berlin, 1993. The language we use is, we hope, descriptive, but it may not appear standard even to experienced geometers.

With the exception of one of the three cubes, these seven models of the Platonic solids all have the characteristic that when they are disassembled they come apart into three or more pieces so that the original polyhedron can be "turned inside out," leaving a cavity in the shape of the original polyhedron. To be more precise, the models are constructed so that the pyramids (whose bases are the faces of the original polyhedron, with height equal to the perpendicular distance from the center of the base face to the center of the polyhedron) can be repositioned pointing outwards, rather than inwards. On some of the models the pyramids are partitioned into parts, apparently so they can be maneuvered properly. We will refer to these models, with the exception of the unusual cube, as *invertible polyhedra*.

There are two features shared by all the invertible models. The first is that the coloring of the original solid destroys the symmetry of the underlying group, reducing the symmetry group of the polyhedron to a cyclic or dihedral group—or worse, to a centrally symmetric figure. However, in each case, when the polyhedron is reassembled in its inverted form its surface is monochromatic so that the entire symmetry group of the underlying polyhedron is again revealed: A_4 for the tetrahedron, S_4 for the cube and octahedron, and A_5 for the dodecahedron and icosahedron. This feature, with the color of the inverted model being always different from the colors of the base model, is very helpful in making the transition from the original to the inverted model.

The second feature shared by all the invertible models is that there is always a piece of the model that forms a rotating ring of tetrahedra or, in just one case, two conjoined rotating rings of tetrahedra. The tetrahedra are, of course, not regular, and the number of them in each ring varies depending on the Platonic solid being inverted.

A third feature, mentioned previously, is that in the more complicated models magnets have been strategically placed inside the pieces. This greatly reduces the frustration of having only two hands with which to hold the various parts in place when making the transition from the original to the inverted model, or vice versa. Despite the effect of the magnets we would advise anyone trying to manipulate the models to do so on a surface with a coefficient of friction at least equivalent to that of a plush carpet, since otherwise it is somewhat tricky to hold any of the rotating rings in the proper position long enough to put the magnetized pieces in place. (Snyder Engineering, the importer and distributor of these models, has available a video tape that demonstrates how the models are taken apart and reassembled. The tape, a poster, and a detailed price list are available for $10 from Snyder Engineering.)

The manufacturer of these remarkable models claims that these are the only possibilities for constructing invertible models of the Platonic solids. Although we cannot say how it might be done otherwise, we are not so sure that this is true. For one thing, the tetrahedron is not the only polyhedron that may be used to construct a rotating ring of polyhedra. Any polyhedron that has opposite edges that are *not* parallel in space may be used to construct a rotating ring. A simple example is the truncated tetrahedron. A less obvious example is the hecca-idecadeltahedron, the convex polyhedron having sixteen equilateral triangles for faces. Perhaps the reader would like to try to find other ways of constructing these invertible models.

We now describe briefly some of the models, with comments about why we found them to be mathematically interesting. We will not try to be comprehensive, nor to describe the tactile pleasure of handling the models since, like so much of mathematics, it is really better to *do it yourself*. As Pólya said, "Mathematics is not a spectator sport!" We will leave most of the mysteries of the models for readers to discover for themselves, either by viewing the video tape devoted to them or, even better, by actually manipulating the models.

The only non-invertible polyhedron in the set is a cube composed of three pieces, credited to Paul Schatz. It is shown in Figure 1. (All of the figures in this review are taken from the poster and are reproduced courtesy of Werkstatt für Platonische Körper, Berlin.) When the two congruent non-convex solid pieces are removed from opposite vertices what remains is a rotating ring of six tetrahedra. When this ring is revolved it can be laid on a flat surface so that its boundary forms a perfect equilateral triangle, with no hole in the center, or it can be maneuvered and laid on a flat surface so that its boundary forms a regular hexagon, with an equilateral triangular hole in its center. It is, of course, not difficult to arrange the ring in space so that you can reassemble the cube by inserting the pieces that originally came from opposite vertices. We are confident that this model, though remarkably simple compared with the other models, still has secrets to reveal.

The second model, due to Franz Sykora (see Figure 2), is of an invertible regular tetrahedron. It too has just three pieces. There are two pieces that when removed from opposite edges leave a rotating ring of eight tetrahedra. Oddly

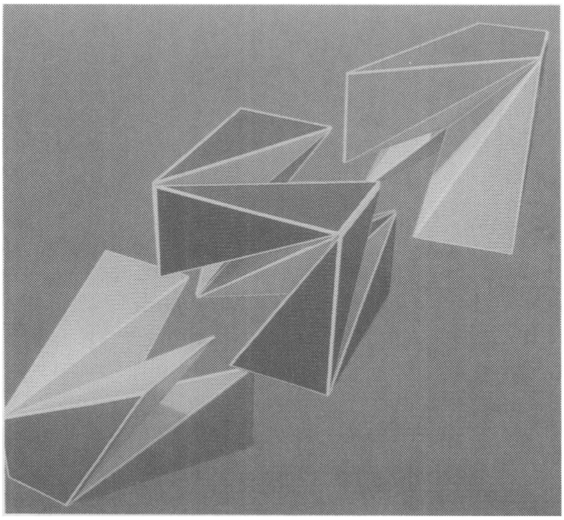


Figure 1

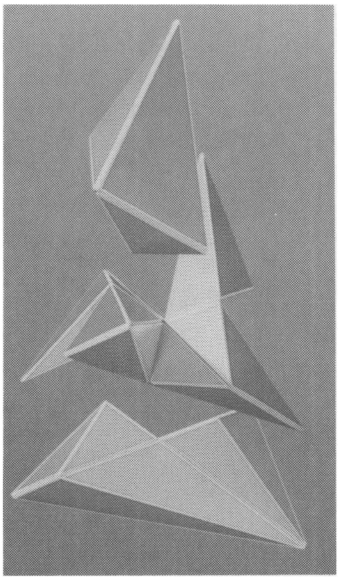


Figure 2

enough we found this model to be the most difficult to reassemble, in both directions. What helped us was remembering that the cavity had to be the regular tetrahedron with which we started, and then we finally had to think of the symmetries involved. The inverted polyhedron is a 12-faced polyhedron, known as the triakis tetrahedron, and has the expected symmetry of the proper rotation group A_4.

The third model, credited to Konrad Schneider (see Figure 3) is of an invertible cube. Again, there are just three pieces. The top and bottom pieces in Figure 3 each consist of a square pyramid with a tetrahedron attached, with cloth hinges, to

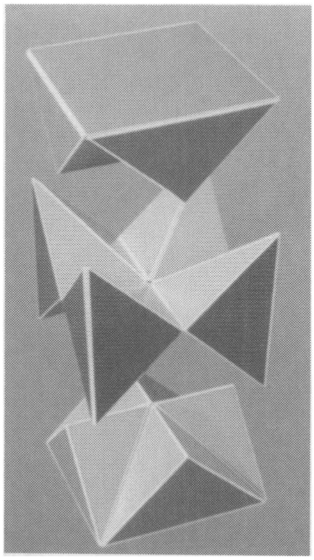


Figure 3

each edge of the base. The third piece is a rotating ring of eight tetrahedra. What is particularly pleasing about this model is that when it is reassembled it forms the rhombic dodecahedron where one can see clearly that the pyramids that sit on the faces of the original cube are just the right height so that the edges of the cube disappear into twelve rhombic faces lying on the edges of the original cube. In fact, the short diagonals of the rhombic faces clearly outline the original cube, if one thinks of how this rhombic dodecahedron sits inside the cubical lattice, and the model shows very vividly why the rhombic dodecahedron must be a space-filler.

The second invertible cube, credited to Wolfgang Maas, has five pieces (see Figure 4): two congruent parts consisting of six tetrahedra, two congruent parts

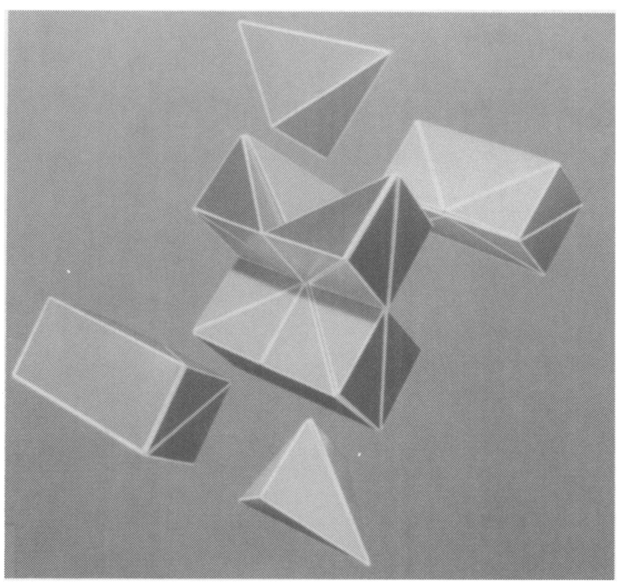


Figure 4

consisting of two tetrahedra, and a single piece consisting of two conjoined rotating rings of eight tetrahedra. The surface of the finished inverted model is, of course, the same as that of the previous cube, although some of its rhombic faces show not just one but two of the diagonals.

The octahedron, credited to Friedemann and Immo Sykora, and the dodecahedron, by Wolfgang Maas, are both composed of just three pieces (see Figures 5 and 6(a)): one piece from each of two opposing faces and a rotating ring that sits between them. For the octahedron the rotating ring is composed of twelve

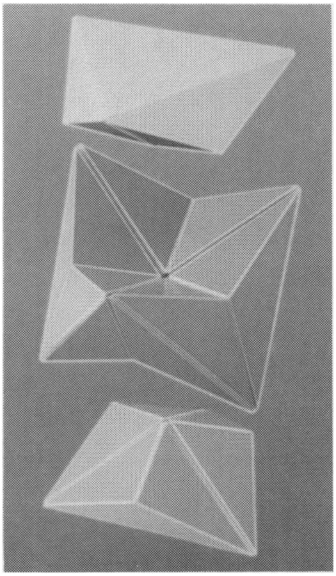


Figure 5

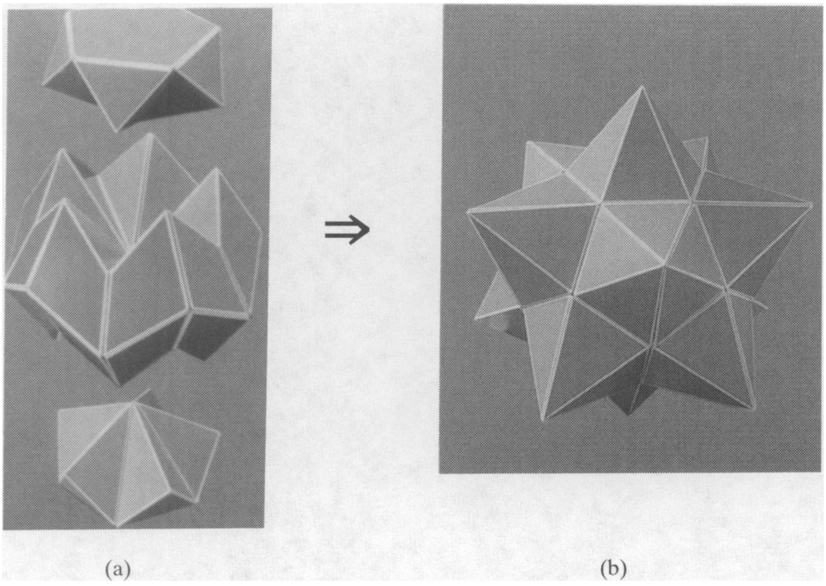
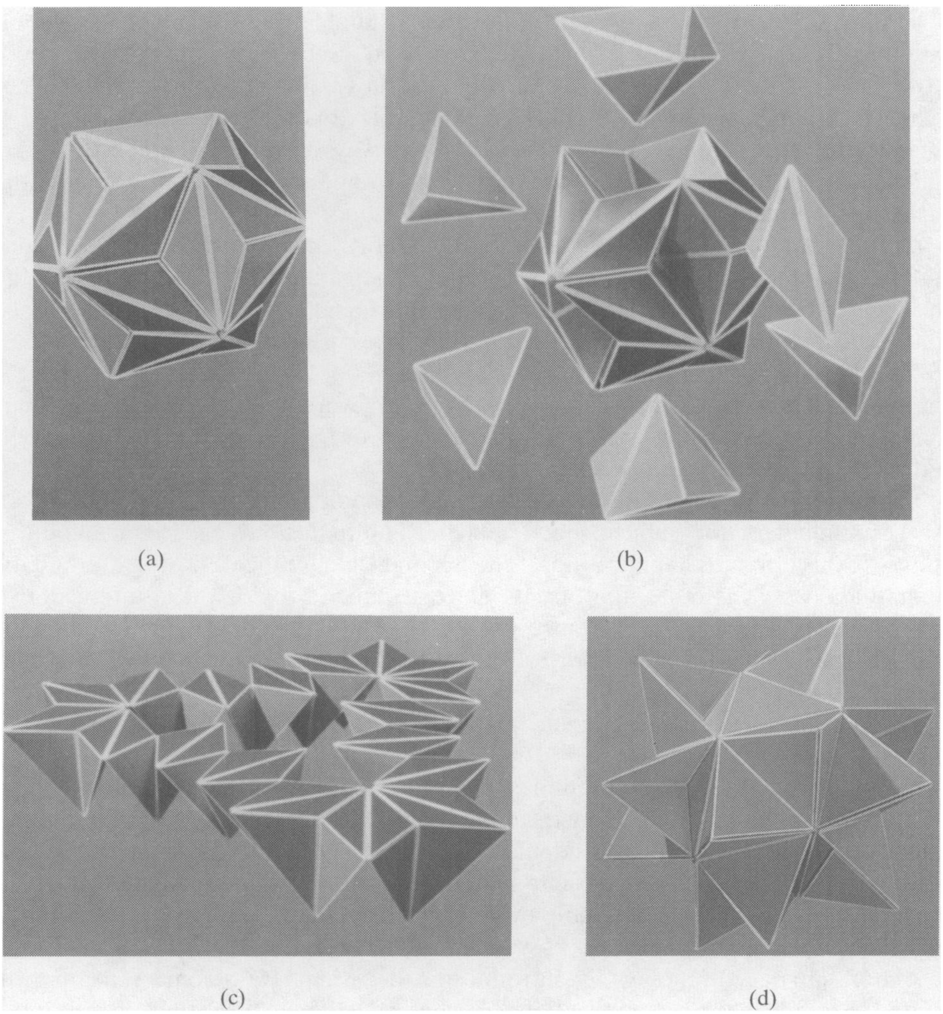


Figure 6

tetrahedra and for the dodecahedron the ring has twenty tetrahedra. Not surprisingly, you can position the rotating rings in interesting ways in the intermediate stages. The resulting inverted polyhedra resemble, but are not quite, the stella octangula (from the octahedron) and the small stellated dodecahedron (from the dodecahedron), as in Figure 6(b). Although their faces are not extensions of the original face planes of the underlying polyhedra they are, nevertheless, beautiful models and they show very vividly, by the removal of one of the smaller pieces, that these are the inversions they claim to be.

The icosahedron, by Immo Sykora, is the most complicated model, consisting of two pieces that are compounds of four tetrahedra, six pieces that are compounds of two tetrahedra, and a spectacular rotating ring of thirty-six tetrahedra (see Figures 7(a), (b), (c)). The model comes with a base icosahedron and a stand, so that you can build the inverted model around it. Without the base model, the stand, and the magnets that are strategically placed within the smaller pieces we would not have been able to assemble the inverted model. With these aids it was



(a) (b) (c) (d)

Figure 7

actually rather easy (well, at least possible!) and immensely satisfying. The resulting model, shown in Figure 7(d), has pyramids on its faces that are too high to be the first stellation of the icosahedron and not high enough to be the great stellated dodecahedron. Despite this minor (and unavoidable!) fault the inverted model is very beautiful to behold.

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