Classification of the homographies in conjugacy classes

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The homographies h are defined in $\mathbb{C}\cup\{\infty\}$ by $z \mapsto \frac{az+b}{cz+d}=Z$ and $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}=Z'$ with $ad-bc\neq 0$.

$\frac{-d}{c}$ (resp. $\frac{-\bar{d}}{\bar{c}}$) $\mapsto \infty$ and $\infty \mapsto \frac{a}{c}$ (if c=0 then ∞ is fixed point : $\infty \mapsto \infty$)

They build a group $\mathcal{H}$ which is generated by the inversions with positive power and the reflexions.
The homographies composed of an even (resp. odd) number of inversions with positive power or reflexions form the sub-group $\mathcal{H}^+$. (resp. the set $\mathcal{H}^-$) and $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$

The most important properties are :

- preservation of angles (orientation is preserved by $h\in\mathcal{H}^+$ and changed by $h\in\mathcal{H}^-$)
- preservation of the set {lines, circles}
- $[z_1,z_2,z_3,z_4]=\frac{z_3-z_1}{z_3-z_2}\div\frac{z_4-z_1}{z_4-z_2}$ is preserved (resp. conjugated) by $h\in\mathcal{H}^+$ (resp. $h\in\mathcal{H}^-$)
- only the reflexions and inversions with positive power, both in $\mathcal{H}^-$, have more than two fixed points

Below a classification of the homographies in conjugacy classes : every homography belongs to one and only one class. The class containing φ is $\mathcal{H}_\varphi=\{h^{-1}\circ\varphi\circ h,\ h\in\mathcal{H}\}$; $\mathcal{H}_{Id}=\{Id\}$.

	generator	**classes – properties**	**fixed points**	**invariant sets**
$\mathcal{H}^+$	central symmetry $Z=-z$	one class which contains $Z=z_o+\frac{\lambda}{z-z_o}$ $\lambda\in\mathbb{C}^*$, $z_o\leftrightarrow\infty$ **involutive**	two : ω_1 and ω_2	two orthogonal pencils defined by ω_1 and ω_2 (base and limit points)
	rotation $Z=e^{i\theta}\cdot z$ $\theta\neq k\pi$	classes indexed by θ (elliptic)	two : ω_1 and ω_2	Poncelet's pencil (limit points ω_1 and ω_2)
	homothecy $Z=r\cdot z$ $r\in\mathbb{R}^*$, $\lvert r\rvert\neq 1$	classes indexed by k (hyperbolic)	two : ω_1 and ω_2	pencil with base points ω_1 and ω_2
	direct similitude $Z=\lambda\cdot z$ $\lambda\in\mathbb{C}^*$, $\lambda\notin\mathbb{R}$, $\lvert\lambda\rvert\neq 1$ (homothecy ∘ rotation, same centre)	classes indexed by λ (loxodromic)	two : ω_1 and ω_2	None
	translation $Z=z+1$	one class (parabolic)	one : ω	pencil with contact point ω
$\mathcal{H}^-$	reflexion $Z'=\bar{z}$	one class which contains the **inversions** with positive power **involutive**	line or circle Γ	all lines and circles $\perp$ to Γ
	inversion $Z'=\frac{-1}{\bar{z}}$ (power<0) (inversion power>0 ∘ centr. symmetry, same centre)	one class : the **inversions** with negative power **involutive**	none	one circle Γ and all circles going through two opposite points of Γ (and z and Z')
	inverse similitude $Z'=r\cdot\bar{z}$ $r\in\mathbb{R}^+$, $r\notin\{0;1\}$ (homothecy ∘ reflexion, centre on the axis)	classes indexed by r	two : ω_1 and ω_2	two, $\perp$ and going through ω_1 and ω_2 a
	glide $Z'=\bar{z}+1$ (reflexion ∘ translation, along the same line)	one class	one : ω	one line or circle going through ω
	$Z'=\frac{e^{i\theta}}{\bar{z}}$ $\theta\neq k\pi$ (inversion ∘ rotation, same centre)	classes indexed by θ	none	one line or circle