The Challenge of Classifying Polyhedra

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The subject of this article is not currently in the mainstream of popular mathematical topics. Nevertheless, teachers of geometry or special classes for liberal arts students may very well want to add this topic to their courses, or use it for an extracurricular offering in a mathematics club. If so, their students will be closer to the frontiers of mathematical research than most.

The material presented was motivated by George Pólya's article, "Guessing and Proving," in the January 1978 issue of *TYCMJ*. Pólya shows there how Euler *probably* proceeded in looking for a classification for polyhedra—but found instead the remarkable relation, $V + F = E + 2$. If you present Euler's formula for polyhedra as outlined there, your students may gain some appreciation for the power of analogy, insight into how guessing and proving relate to scientific work in progress, and more comprehension of the significance of the results in this article.

Because of the sequential aspects of this article, we will not discuss or redefine in detail all of the terms used in Pólya's article. We will begin with an example from that article and continue to use the general approach of "guessing and proving," but our attention will be focussed on Euler's original question: *How can polyhedra be classified so that the result is in some way analogous to the simple classification of polygons according to the number of their sides*? Here we make some progress and find another formula. The question is not entirely settled, but perhaps after studying this article you, or some of your students, may be able to contribute the next important link, or even complete the solution of this classical problem.

How Do You Classify Combinatorially Distinct Polyhedra?

Consider this guess: If you know the number of faces, F, vertices, V, and edges, E for any simply connected polyhedron (of genus 0), then the three numbers, F, V, E may serve to classify such polyhedra.

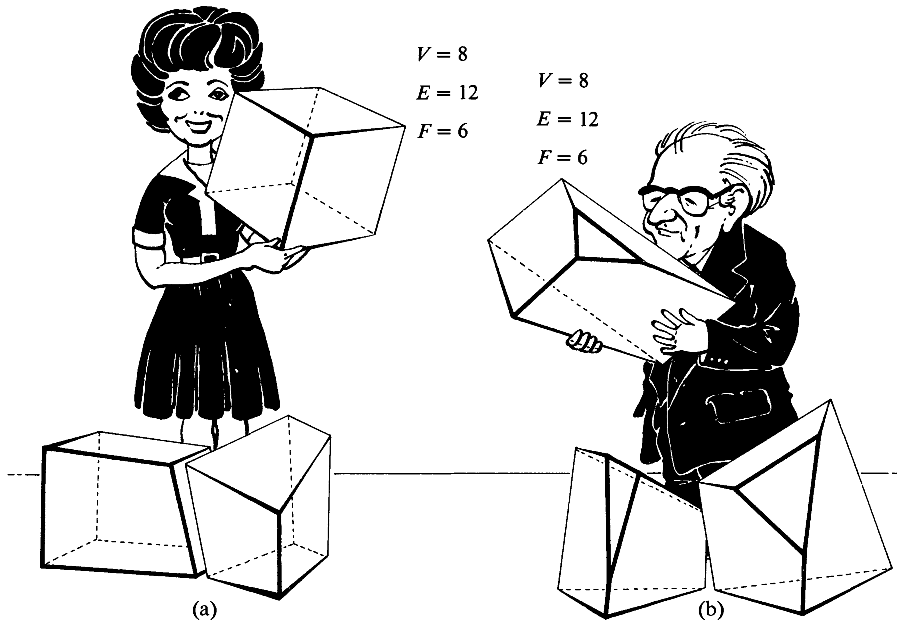


Figure 1.

Now look at Figure 1 in which all members of both sets of polyhedra have the same values for F, V and E respectively. We would be willing to let any of the polyhedra in Figure 1(a) belong to the same class, and likewise for the collection of polyhedra in Figure 1(b). But the combinatorial properties of any one of the polyhedra in the first set are so essentially ("morphologically") different from any of the ones in the second that they cannot be put in the same class. So, we conclude that this guess is unreasonable.

The featured polyhedra in Figure 1 were the final pair in a carefully selected sequence of polyhedra in Pólya's article, and I originally thought I could see how to distinguish between them—in a way that had not been mentioned there. That idea had very limited success (as was predicted immediately by Pólya)—so let us not resurrect it. I mention it only to comfort others who may have similar experiences in attacking this problem.

Then Professor Pólya suggested the following:

> On a given convex polyhedron, let
>
> f_s = the number of faces with s sides;
>
> v_s = the number of s-edged vertices;
>
> and then denote the polyhedron by the infinite sequence:
>
> $$f_3, f_4, f_5, \ldots ; v_3, v_4, v_5, \ldots,$$
>
> in which, however, there are only a finite number of non-zero terms.

According to this scheme the cube would be denoted "0, 6; 8" and the "cheese" would be "2, 2, 2; 8." This was encouraging. It appeared that knowing the number and nature of all the faces, and the number and nature of all the vertices *might* be sufficient for the desired classification. (Let your students look for counterexamples before going on.)

But alas! This didn't always work either. Pólya found the counterexample represented by the pair of polyhedra in Figure 2.[1] Intuitively, one senses that these two polyhedra are combinatorially distinct. Each triangle in 2(a) is intimately connected with some other triangle, while the triangles in 2(b) seem, by comparison, to be mere acquaintances of each other (in 2(a) each triangle, and in 2(b) no triangle, has a common side with another triangular face). Yet, each is composed of 4 triangular faces, 4 quadrilateral faces, and each has 4 three-edged and 4 four-edged vertices. Indeed, using the proposed scheme, the name of each would be "4, 4; 4, 4."

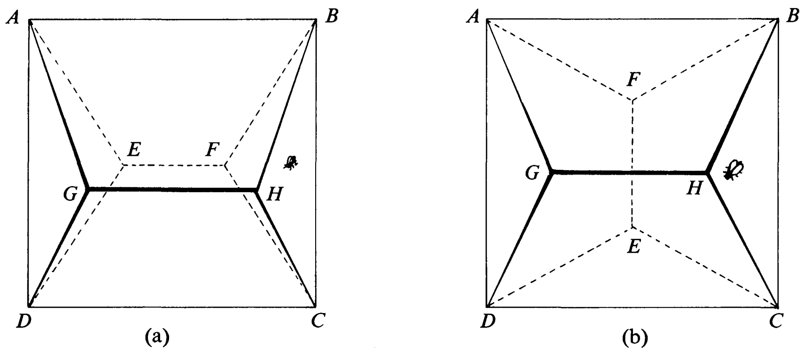


Figure 2.

What a pity! Then again, perhaps we can learn from this counterexample and devise another scheme which *may* work. What we need is to *formulate* some way in which the two polyhedra in Figure 2 are different from each other.

With this objective in mind, suppose we place a nearsighted flea on a triangular face of the polyhedron in Figure 2(a) and a nearsighted fly on a triangular face of the polyhedron in Figure 2(b). Then instruct them to walk around the boundary of the face on which they were placed and count the faces that adjoin (have a common edge or vertex with) that face. Both of these nearsighted insects will count 5 adjoining faces. But suppose we find a flea and a fly who are both farsighted and place them, as before, on triangular faces of the polyhedra in Figure 2(a) and 2(b) respectively. This time instruct each of them to walk near the boundary of the face on which they were placed and count the non-adjoining *sides* of the faces that adjoin the face on which they walk. The flea on face HBC will now count 6 "surrounding sides" (AE, EF, FE, ED, DG, GA) and the fly on face HBC will count 5 "surrounding sides" (AF, FE, ED, DG, GA). This is the clue we were looking for. It gives us a means by which we can attach a number to each face of the

[1] Subsequently, Professor Pólya observed that there are many counterexamples and you could find them listed conveniently in the article by P. J. Federico referred to below. There are no counterexamples with six faces or fewer.

polyhedron. In Figure 2(a) the four triangular faces would all be labeled 6 and the four quadrilateral faces would all be labeled 4. But, in Figure 2(b) all faces would be labeled 5.

Voilà! We can now see a general plan emerging. It involves taking a look at various surrounding properties of parts of the geometric configurations we seek to classify. In the case of convex polygons in the plane, a blind insect walking around the boundary, feeling the number of sides, would be sufficient. In the case of polyhedra with six faces or less a very nearsighted insect could describe the polyhedron satisfactorily for us, in terms of the number and nature of its faces and vertices. Intuition suggests that in dealing with polyhedra having more faces, you must look farther away from the various parts in order to insure enough interconnectedness in the data to obtain classification. Because this seems so promising, we will formalize it.

First, however, notice that farsighted insects don't perceive the non-adjoining boundaries of the adjoining faces as edges of a polyhedron—because they can't see around corners. Thus we take the viewpoint that if a line segment joining two vertices is considered in relation to a face, to whose boundary it belongs, it is called a *side* of that face. If it is considered in relation to the whole polyhedron, as the common boundary of two neighboring faces, it is called an *edge* of that polyhedron. We could, of course, take a similar view with regard to corners (on faces) and vertices (on polyhedra), but we do not plan to count either of these entities, so it would be a pointless distinction. We will, however, make the following:

DEFINITIONS

1. A *Convex Polyhedron* is (from the standpoint of combinatorial topology) equivalent to a sphere.
2. The *Constituent Parts* of a convex polyhedron are of three kinds:

   vertices, edges, faces,

   conceived of as closed sets of points.
3. Two different constituent parts (of the same kind or not) are called *adjoining* if they have at least one common point. (They may have an infinity of common points.)
4. "*A* adjoins *B*"
"*B* adjoins *A*"
"*A* and *B* are adjoining" } all mean the same thing and

   in particular:

   "Face adjoining face" means "one common vertex or side (meeting on an edge of the polyhedron)";

   "face adjoining vertex" means "a vertex of the polyhedron is also a vertex of the face";

   "edge adjoining edge" means "the two edges have ONE endpoint in common";

   "edge adjoining vertex" means "the vertex is an endpoint of the edge."

   Moreover, a vertex NEVER adjoins another vertex.
5. Each constituent part is classified according to its number of *surrounding sides*; that is, the number of non-adjoining sides on the adjoining faces to that part.

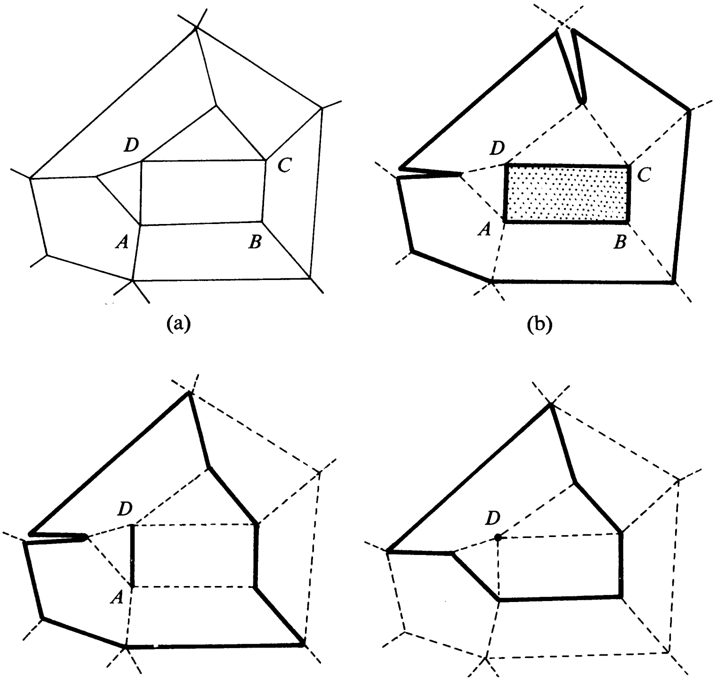


Figure 3.

Notice that the definitions give us a chance to look not only at the surroundedness of the faces, but the surroundedness of the vertices and edges as well. Thus, for example, a nearsighted insect placed on quadrilateral *ABCD* in Figure 3(b) (showing a portion of some polyhedron) would count 10 surrounding sides for that face. If placed on the edge *AD* he would count 10 surrounding sides, and if he were placed on vertex *D* he would count 7 surrounding sides. The question is: Will we need *all* of this information? Perhaps the memory of our previous limited successes is affecting our judgment—but if it turns out that we now get more information than needed, we can delete it later. So let us give it ANOTHER TRY.

> On a given polyhedron, let each of its constituent parts be classified according to its number of surrounding sides, s (obtained by counting the non-adjoining sides of the adjoining faces to that constituent part).
> Let V_s = the number of vertices with s surrounding sides,
> E_s = the number of edges with s surrounding sides,
> F_s = the number of faces with s surrounding sides;
> and then denote the polyhedron which generated these numbers by the infinite sequence:
>
> $$V_3V_4V_5, \ldots ; E_2E_3E_4, \ldots ; F_0F_1F_2, \ldots ,$$
>
> in which, however, there are only a finite number of non-zero terms.

Note that there are no commas between successive entries of the same kind in the sequence. Entries with more than one digit will be written in parentheses.

At this point it is worthwhile to have your students compute this sequence of classifying numbers for some familiar polyhedra. As an example, let us consider the cube. In Figure 4(a) we see that the quadrilateral face *ABCD* has four surrounding sides (*EF*, *FG*, *GH*, *HE*). By symmetry every other face must also have four surrounding sides. Likewise, since edge *BC* has six surrounding sides (*AD*, *DH*, *HG*, *GF*, *FE*, *EA*) all edges will have six surrounding sides. Finally, since vertex *F* has six surrounding sides (*AE*, *EH*, *HG*, *GC*, *CB*, *BA*) all vertices will have six surrounding sides. Thus the sequence of numbers "0008; 0000(12); 00006" will denote the cube (and the other polyhedra shown in Figure 1(a)).

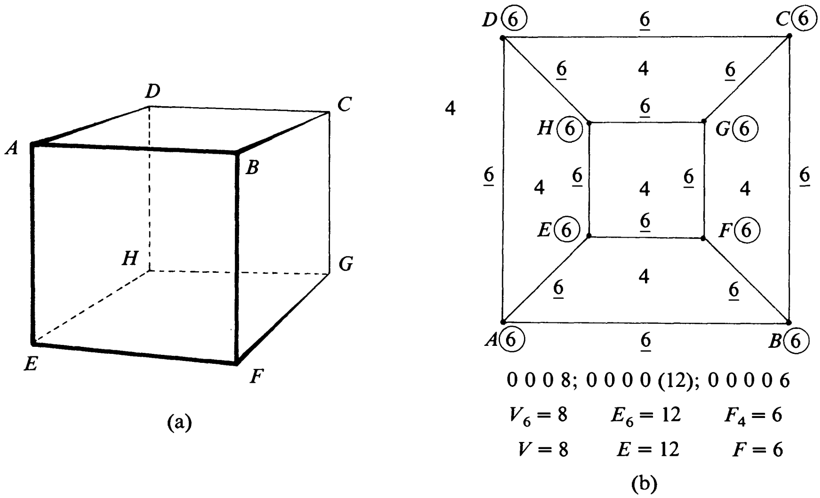


Figure 4.

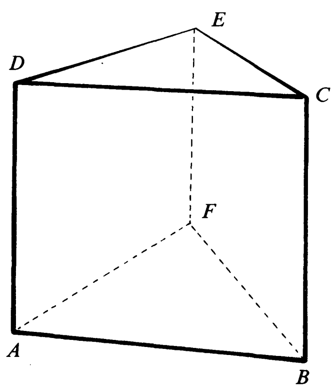
Table I

(for Figure 4).

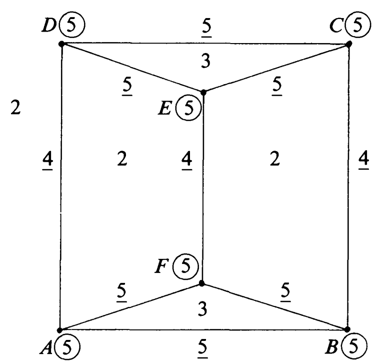
Constituent Part	Surrounding Sides	Classifying Number
Quadrilateral *ABCD*	*EF*, *FG*, *GF*, *HE*	4
Quadrilateral *ABFE*	*DC*, *CG*, *GH*, *HD*	4
Quadrilateral *EFGH*	*AB*, *BC*, *CD*, *DA*	4
Edge *AB*	*DH*, *HE*, *EF*, *FG*, *GC*, *CD*	6
Edge *BF*	*AE*, *EH*, *HG*, *GC*, *CD*, *DA*	6
Edge *EF*	*AD*, *DH*, *HG*, *GC*, *CB*, *BA*	6
Vertex *A*	*DH*, *HE*, *EF*, *FB*, *BC*, *CD*	6
Vertex *E*	*DH*, *HG*, *GF*, *FB*, *BA*, *AD*	6

It is difficult to show all of the classifying numbers on the constituent parts of the illustration in Figure 4(a), so we use instead the Schlegel diagram of the cube shown in Figure 4(b). To see the relationship between the two figures, imagine the face *ABCD* to be transparent. Then view the cube at extremely close range through the face *ABCD*. The edges will then be perceived as shown in Figure 4(b). As can be seen on the resulting figure it is now easy to show all of the classifying numbers. One must be careful to remember that the boundary represents a face, and each of the separate regions within that boundary also represents a face. Once the classifying numbers are assigned to the constituent parts it is then possible to read the classifying sequence from the diagram. If you use different colors for the face, edge and vertex numbers, reading the sequence is even easier. Since colors are not possible here, the circled numbers represent the surrounding sides for vertices, and the underlined numbers represent the surrounding sides for the nearest edge. The remaining numbers represent the surrounding sides for the face corresponding to the region in which they appear, except for the number outside of the bounding polygon (on the left), which represents the surrounding sides for the bounding polygon.

With practice you can compute the classifying numbers for all the constituent parts of a polyhedron using only the Schlegel diagram. But it helps to begin with a familiar configuration. Notice that on the Schlegel diagram for the cube, the faces and edges appear in three different ways and the vertices appear in two different ways. It may be helpful to identify on the Schlegel diagram the surrounding sides for each of the cases listed on Table I.



(a)



0 0 6; 0 0 3 6; 0 0 3 2

$V_5 = 6$ $E_4 = 3$, $E_5 = 6$ $F_2 = 3$, $F_3 = 2$

$V = 6$ $E = 3 + 6 = 9$ $F = 3 + 2 = 5$

(b)

Figure 5.

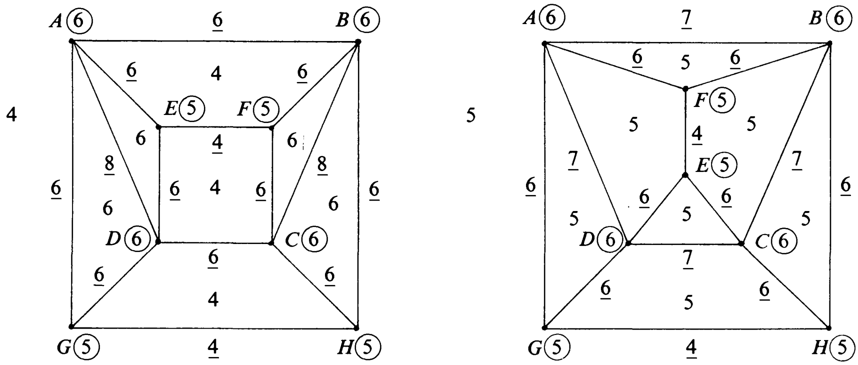
Figure 5 represents a particularly instructive example. You may wish to draw the Schlegel diagram first and try to attach the various classifying numbers yourself. Table II may be useful.

Table II

(for Figure 5).

Constituent Part	Surrounding Sides	Classifying Number
Triangle *ABF*	*DE*, *EC*, *CD*	3
Quadrilateral *ABCD*	*EF*, *FE*	2
Quadrilateral *FBCE*	*DA*, *AD*	2
Edge *AB*	*DE*, *EF*, *FE*, *EC*, *CD*	5
Edge *AF*	*DE*, *EC*, *CB*, *BC*, *CD*	5
Edge *EF*	*AB*, *BC*, *CD*, *DA*	4
Edge *DA*	*CE*, *EF*, *FB*, *BC*	4
Vertex *A*	*DE*, *EF*, *FB*, *BC*, *CD*	5
Vertex *F*	*DE*, *EC*, *CB*, *BA*, *AD*	5

We return now to the polyhedra in Figure 2. The appropriate Schlegel diagrams (each viewed through face *AGHB*) are shown in Figure 6 with the classifying sequence and certain data obtainable from that sequence.



(a)

0 0 4 4; 0 0 2 0 (10) 0 2; 0 0 0 0 4 0 4

$V_5 = 4$	$E_4 = 2$	$F_4 = 4$
$V_6 = 4$	$E_6 = 10$	$F_6 = 4$
	$E_8 = 2$	
$V = 8$	$E = 14$	$F = 8$
$\sum_{s=3}^{6} sV_s = 44$	$\sum_{s=2}^{8} sE_s = 84$	$\sum_{s=0}^{6} sF_s = 40$

(b)

0 0 4 4; 0 0 2 0 8 4; 0 0 0 0 0 8

$V_5 = 4$	$E_4 = 2$	$F_5 = 8$
$V_6 = 4$	$E_6 = 8$	
	$E_7 = 4$	
$V = 8$	$E = 14$	$F = 8$
$\sum_{s=3}^{6} sV_s = 44$	$\sum_{s=2}^{7} sE_s = 84$	$\sum_{s=0}^{5} sF_s = 40$

Figure 6.

The classifying sequences in Figure 6 suggest that we might need only the portion of the sequence involving the edges, or the faces. But the data shown on Table III (for the polyhedra represented in Figure 7) verify that we need the entire sequence of numbers to classify those polyhedra. In view of Table III it is quite remarkable that this sequence does indeed classify all combinatorially distinct polyhedra with eight faces or fewer.[2]

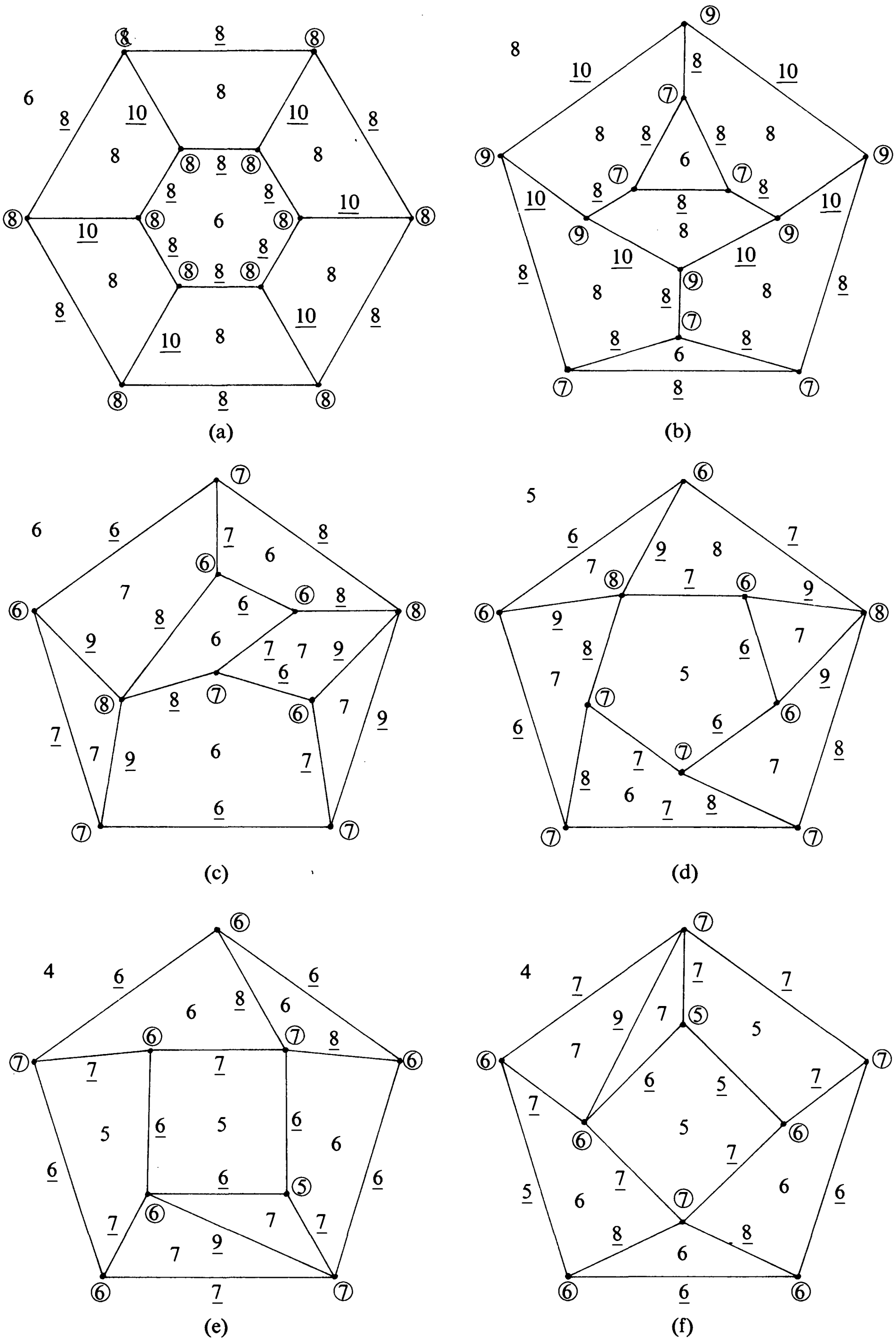


Figure 7.

[2] All of the 301 polyhedra with 8 faces or fewer are listed and illustrated by Schlegel diagrams in Federico's paper which was referred to in order to verify that this scheme does classify those polyhedra. The portion of the sequence involving edges, "$E_2E_3E_4, \ldots,$" will classify those polyhedra with 7 faces or less, but the entire expression is needed when you consider polyhedra with more than 7 faces.

Table III.

Fig. No.	Federico's Fig. No.	Classification Sequence											
		V_3	V_4	V_5	V_6	V_7	V_8	V_9;	E_2	E_3	E_4	E_5	E_6
7(a)	54	0	0	0	0	0	(12)	0;	0	0	0	0	0
7(b)	57	0	0	0	0	6	0	6;	0	0	0	0	0
7(c)	159	0	0	0	4	4	2	0;	0	0	0	0	4
7(d)	162	0	0	0	4	4	2	0;	0	0	0	0	4
7(e)	234	0	0	1	5	3	0	0;	0	0	0	0	7
7(f)	235	0	0	1	5	3	0	0;	0	0	0	2	3

Classification Sequence (continued)												
E_7	E_8	E_9	E_{10};	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
0	(12)	0	6;	0	0	0	0	0	0	2	0	6
0	(12)	0	6;	0	0	0	0	0	0	2	0	6
4	4	4	0;	0	0	0	0	0	0	4	4	0
4	4	4	0;	0	0	0	0	0	2	1	4	1
5	2	1	0;	0	0	0	0	1	2	3	2	0
7	2	1	0;	0	0	0	0	1	2	3	2	0

Table IV.

Fig. No.	$V = \sum V_s$	$E = \sum E_s$	$F = \sum F_s$	Sum of vertex numbers $= \sum sV_s$	Sum of edge numbers $= \sum sE_s$	Sum of face numbers $= \sum sF_s$
7(a)	12	18	8	96	156	60
7(b)	12	18	8	96	156	60
7(c)	10	16	8	68	120	52
7(d)	10	16	8	68	120	52
7(e)	9	15	8	56	102	46
7(f)	9	15	8	56	102	46

For polyhedra with more than 8 faces the question is still not resolved (no counterexamples have been found). Thus, there remain

Some Questions

Will this sequence classify all combinatorially distinct polyhedra? If so, how is it proved? If not, what is a counterexample?

A Surprising Bonus

There is little doubt that the proposed classification scheme is somewhat tedious, and a simpler one would certainly be preferred.[3] Meanwhile, some sort of check on the accuracy of the labels obtained here is certainly desired. Of course, you might check to make certain

$$F + V = E + 2, \quad \text{where} \quad F = \sum_{s=0}^{\infty} F_s, \; V = \sum_{s=3}^{\infty} V_s \quad \text{and} \quad E = \sum_{s=2}^{\infty} E_s.$$

But this is not a very strong test, for it simply assures that it is unlikely you have forgotten to assign a number to one, or more, of the constituent parts. What we really want is a test that will give you some reasonable assurance that you haven't miscounted the surrounding sides for some constituent part.

What can we use? Perhaps, if we are lucky, something *like* Euler's formula might work. Let's look at the various classifying numbers on the cube. Each of its six faces is labeled 4; each of its eight vertices is labeled 6, and each of its twelve edges is labeled 6 (in more abbreviated form: $F_4 = 6$, $V_6 = 8$, $E_6 = 12$).

Of course we know that (6)(4) + (8)(6) = (12)(6).

Do you suppose that

$$\begin{Bmatrix}\text{sum of all}\\ \text{classifying}\\ \text{FACE numbers}\end{Bmatrix} \quad \text{plus} \quad \begin{Bmatrix}\text{sum of all}\\ \text{classifying}\\ \text{VERTEX numbers}\end{Bmatrix} \quad \text{equals} \quad \begin{Bmatrix}\text{sum of all}\\ \text{classifying}\\ \text{EDGE numbers}\end{Bmatrix}?$$

To put it more formally, do you suppose that

$$\sum_{s=0}^{\infty} sF_s + \sum_{s=3}^{\infty} sV_s = \sum_{s=2}^{\infty} sE_s,$$

is true for all convex polyhedra? (Let your students examine more cases, see Table IV, before giving away this final result.)

Happily, the answer is YES. To see why this should be so, focus your attention on one side of some n-gon on a polyhedron and see how many times that particular side gets counted by faces, by vertices and by edges. In Figure 8 let k represent the number of times the indicated side gets counted by faces adjoining the n-gon. Then $k + 1$ is the number of times that side gets counted by the edges radiating from the vertices of the n-gon which are not on the boundary of the n-gon. But, the side under consideration also gets counted $n - 3$ times by edges on the boundary of the n-gon. Finally, this side gets counted $n - 2$ times by vertices on the n-gon. Thus, we see that this particular side gets counted

k times by faces,

$n - 2$ times by vertices,

[3] When I presented this material at California State College, Sonoma, a student, Rick DeFreez, suggested the following addition to Professor Pólya's scheme: Add to $f_3, f_4, f_5, \ldots, v_3, v_4, v_5, \ldots,$ as a complement, the sequence ${}_3e_3, {}_3e_4, {}_3e_5, \ldots,$ where ${}_ie_j$ is the number of edges adjoining an i-sided face and a j-sided face. Mr. DeFreez's idea does, in fact, distinguish between Figures 2(a) and 2(b) but, again, it has limited success in general (see, for example, Figures 282 and 288 in Federico's paper). Nevertheless, it was a good try; perhaps something similar may be found that will have fewer limits to its success.

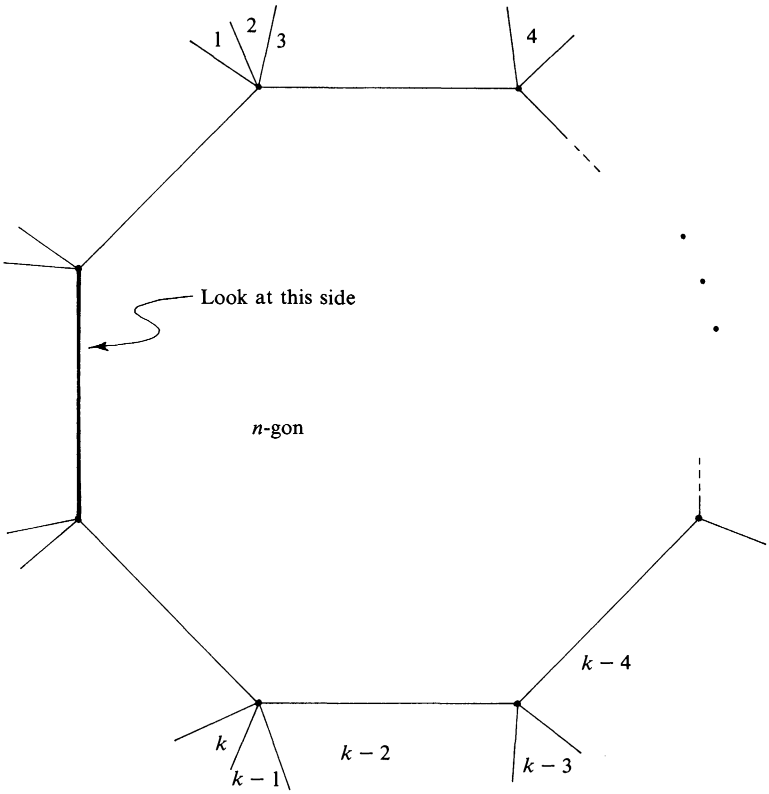


Figure 8.

and

$$(k + 1) + (n - 3) \text{ times by edges.}$$

But certainly, $k + (n - 2) = (k + 1) + (n - 3)$, and since a similar relationship holds for each of the $2E$ sides on the polyhedron, it follows that this new Euler-like relationship must hold.

This formula, along with Euler's formula, provides a reasonable check for the accuracy of a classifying sequence.

I express my sincere thanks to Professor Pólya for his continued interest in discussing the problem; for encouraging me to write this article; for reading the preliminary version and especially for his valuable suggestions with regards to the definitions.

I also thank Dave Logothetti for his encouragement and for his creative illustrations.

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