The Bellows Conjecture

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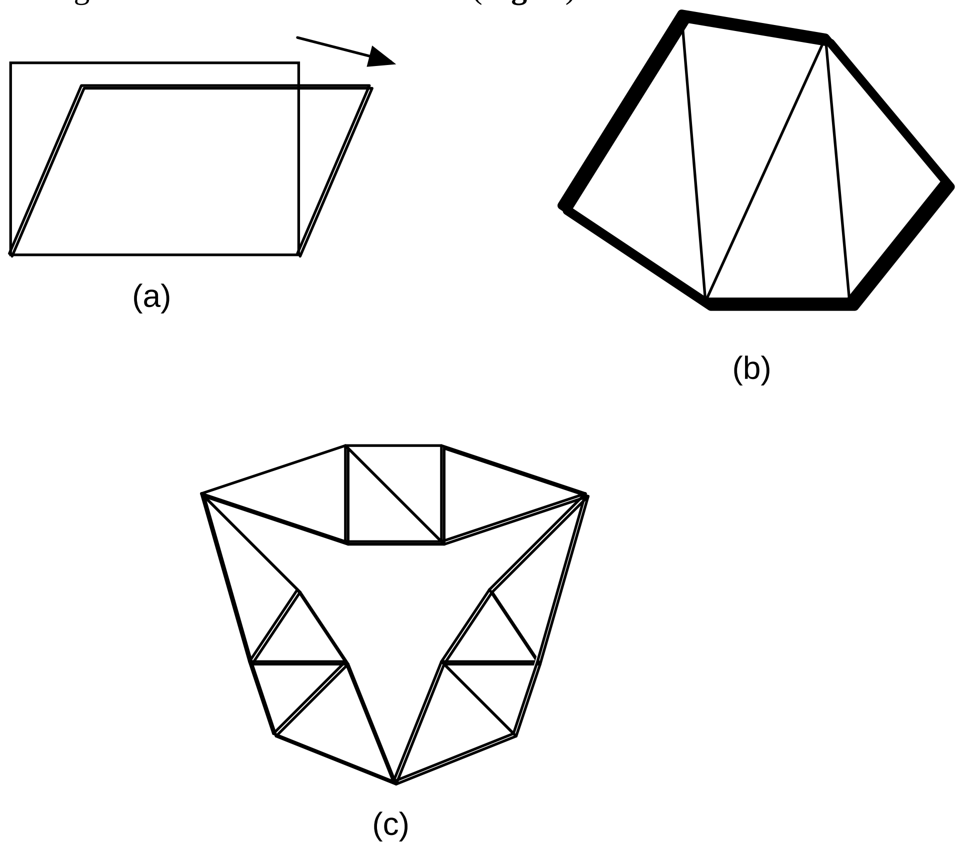
Every amateur carpenter who has tried to build a bookcase knows that rectangles are not rigid. If you lean against the corner of a rectangle then it tilts sideways to form a parallelogram (**Fig.1a**) --- and in all likelihood collapses completely. A triangle, on the other hand, *is* rigid: it cannot be deformed without changing the length of at least one side. Euclid knew this, in the form 'if two triangles have sides with the same lengths, then the triangles are congruent (have the same shape)'. In fact, the triangle is the *only* rigid polygon in the plane. Any other polygonal shape must be braced in some manner. For example, cross-struts can be added, to break it into triangles (**Fig.1b**), or shapes that are themselves rigid can be assembled in threes (**Fig.1c**).



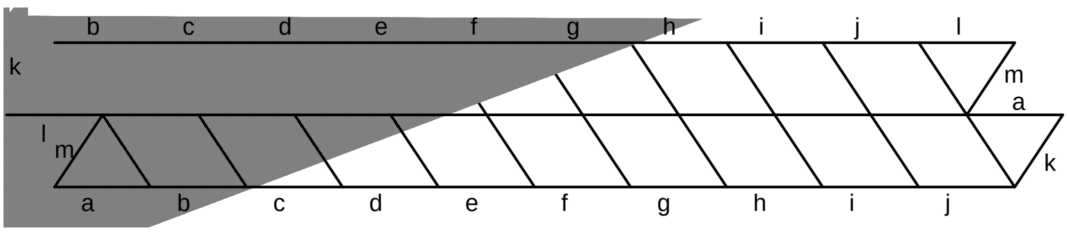
(a) Flexing a rectangle changes its area. (**b**) Cross-struts can make a polygon rigid. (**c**) Some rigid shapes need not be made from triangles.

Another way to rigidify your bookcase is to nail a flat back onto it. This takes the question into the third dimension, where everything becomes far more interesting, and surprises abound. For nearly 200 years mathematicians have been puzzled by the rigidity, or otherwise, of polyhedrons --- solids with finitely many straight line edges, whose faces are polygons that meet in pairs along edges. Until recently it was assumed that any polyhedron with triangular faces must be rigid --- but that turned out not to be true. There exist 'flexible' polyhedrons, which change shape even though no face distorts or bends by even the tiniest amount --- I'll come back to those in a moment.

The latest discovery, made by Robert Connelly (Cornell), Idzhad Sabitov (Moscow State U) and Anke Walz (Cornell), is that flexible polyhedrons cannot change their volume. It is not possible to make a polyhedral 'bellows' that can flex and blow air out through a hole as its internal volume shrinks. (What about concertinas? See below.) Their proof required them to discover some unexpected properties of polyhedrons, which are likely to prove important in future research.

Before starting on the math, I'd better make one thing clear. Anyone who has folded origami figures from paper knows that it is possible to make birds that flap their wings, frogs whose legs move, and so on. Aren't these flexible polyhedrons? The answer is 'no', for two reasons. One reason is that the paper has edges, so it does not form a polyhedron. The other, more important, reason is that when the paper frog moves its legs, the paper *bends* slightly. The same goes for concertinas, which at first sight appear to be polyhedral bellows, but again these work only because of slight bending (and perhaps even a little stretching). From now on, no amount of bending, not even by a billionth of an inch, will be permitted. When a polyhedron flexes, the only things that can change are the angles at which faces meet. Imagine that the faces are hinged along their edges, and flex the hinges. All else is perfectly rigid.

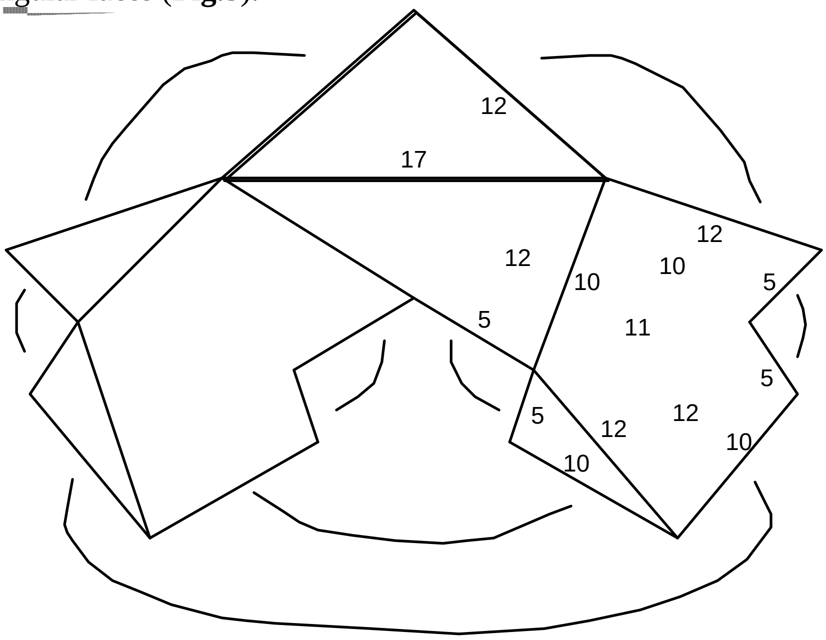
The whole area dates back to 1813, when the great French mathematician Augustin Louis Cauchy proved that a convex polyhedron --- one without indentations --- cannot flex. But what if there are indentations? The first flexible non-convex polyhedron was found by Raoul Bricard, a French engineer --- except that in his example faces were permitted to interpenetrate freely, and move through each other. This is of course impossible for a real physical object. However, Bricard's example can be realised if we remove the faces and replace the edges by rigid rods to get a *linkage*. Bricard also invented chains of simple polyhedrons, joined edge to edge, that can flex. According to W.W. Rouse Ball's famous book *Mathematical Recreations and Essays,* the simplest such rings were invented by J. M. Andreas and R.M.Stalker [EDITOR: NO FURTHER INFORMATION GIVEN OR KNOWN FOR BALL, ANDREAS, STALKER. DATE NOT KNOWN. BALL IS MODERATELY FAMOUS, THE OTHER TWO OBSCURE.]. These are rings of six or more regular tetrahedrons --- the number must always be even --- hinged together along pairs of opposite edges (**Fig.2**).



A ring of ten tetrahedrons. Fold solid lines into ridges, dotted ones into valleys. Join tabs with the same letter.

With six tetrahedrons the amount of movement is slight, but with eight or more, the ring can rotate indefinitely, like a smoke ring. With 22 or more, the ring can even be knotted! However, such shapes are not true polyhedrons because more than two faces meet along some edges.

The topic did not really come alive until the 1970s, when Connelly modified Bricard's self-penetrating flexible polyhedron in such a manner that it remained flexible, but ceased to be self-penetrating. Within a few years the construction had been simplified by Klaus Steffen (U Düsseldorf), to yield a flexible polyhedron with nine vertices and fourteen triangular faces (**Fig.3**).



Steffen's flexible polyhedron. Fold solid lines into ridges, dotted ones into valleys.

It is amusing to make a model out of thin card, and see how it flexes. As far as anyone knows, this is the simplest possible flexible polyhedron, but it is very difficult to see how to go about proving such a statement.

Mathematicians who investigated these and other flexible polyhedrons quickly noticed that as they flexed, some parts moved closer together while others moved further apart. Qualitatively, at least, it looked as if the total volume might not change during the motion. Dennis Sullivan (City U of New York) filled a flexible polyhedron with smoke, flexed it, and observed that no smoke puffed out. This elegant but crude experiment suggested --- but of course did not prove --- that the volume remained unchanged. And so the Bellows Conjecture was born. It states that a flexible polyhedron has constant volume while it flexes --- a polyhedral bellows is impossible.

The first interesting feature of the Bellows Conjecture is that its planar analog is false. When a flexible polygon, such as a rectangle, collapses into a parallelogram, the area gets smaller. Clearly there is something unusual about three-dimensional space that makes a bellows impossible. But what? Connelly's group focussed on a famous formula for the area of a triangle, believed to be due to Archimedes, but usually credited to Heron of Alexandria who wrote down a proof. Heron was a Greek mathematician who lived somewhere between 100 BC and AD 100, and he stated and proved the formula in his books *Dioptra* and *Metrica*. The formula is shown in the box, but what matters here is not so much the details, as the general nature of the formula. It can be rearranged, using algebra, to give an equation relating the area of the triangle to its three sides. Moreover, this equation is polynomial: its terms are just whole number powers of the variables, multiplied by fixed numbers.

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BOX Heron's Formula

Suppose that a triangle has sides a, b, c, and area x. Let s be the semi-perimeter:

$$s = (a+b+c)/2.$$

Then

$$x = \sqrt{s(s-a)(s-b)(s-c)}.$$

Square this equation and rearrange to get rid of the 1/2's: the result is

$$16x^2 + a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = 0.$$

This is a polynomial equation relating the area x to the three sides a, b, c.

=====END BOX

Sabitov came up with the curious --- and at first implausible --- idea that there might be a similar polynomial equation for *any* polyhedron, relating the polyhedron's volume to the lengths of its sides. Such a polynomial would be a truly remarkable discovery, because until that moment nobody had suspected that any such thing could possibly exist. Yes, there were some well known special formulas --- easy ones for for cubes and rectangular boxes, and something a bit like Heron's formula for tetrahedrons (solids with four triangular faces, regular or irriegular pyramids on a triangular base) only more messy. But nothing completely general, applying to any polyhedron.

Could the great mathematicians of the past really have missed such a wonderful idea? It seems unlikely.

Nevertheless, suppose such a formula *does* exist. Then the Bellows Conjecture is a simple consequence. The reason is straightforward. The formula relates the volume to the sides. As the polyhedron flexes, the lengths of its sides don't change --- so the formula stays exactly the same. Its solution, the volume, must therefore also stay the same.

Actually, there is one technical point to take care of. A polynomial equation can have *several* distinct solutions, so in principle the volume might suddenly jump from one solution to a different one. However, the volume obviously changes gradually if the flexing is gradual, so whatever the volume does, it cannot jump. End of proof.

All that remained was to prove the existence of a polynomial equation for the volume of a polyhedron in terms of its sides. There is an obvious place to start: the classical formula for the volume of a tetrahedron. Just as any polygon can be divided into triangles, so can any polyhedron be divided into tetrahedrons. Then the volume of the polyhedron is just the sum of the volumes of those tetrahedral pieces. But that won't, of itself, solve the problem. The formula that it leads to involves all the edges of all the pieces, and many of those are not edges of the original polyhedron. Instead, they are various 'diagonal' lines that cut across from one corner of the polyhedron to another one, whose lengths may very well change if the polyhedron flexes. So somehow the formula has to be massaged, algebraically, to get rid of unwanted edges and 'glue' all the component equations together into one Grand Unified Equation.

It was always going to be messy. For an octahedron, with eight triangular faces, it turned out that such a massaging procedure was possible, but the resulting equation involved the 16th power of the volume. More complex polyhedrons would surely require higher powers still. However, the octahedron was a good start. By 1996, Sabitov could write down an explicit but extremely complicated procedure for finding suitable equations. In 1997, however, the team of Connelly, Sabitov, and Walz found a far simpler way to achieve the same result.

The reasons why such equations exist are not fully understood. In two dimensions, they don't --- except for the rigid triangle and the Heron equation. In three dimesions, we now know that they do. Connelly and Walz think they know how to prove a four-dimensional Bellows Conjecture. For five dimensions or more, the problem is wide open. But it's fascinating to see how a simple experiment with some bits of card and some smoke opened up a marvellous, totally unexpected, and fundamental mathematical discovery.