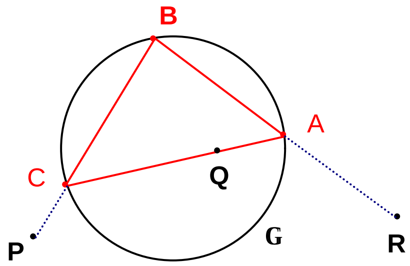
ABSTRACT

Castillon’s problem

A circle G and three points P, Q and R (not on G) are given. Construct all triangles ABC inscribed in G and such as R Î (AB), P Î (BC) and Q Î (CA).

Proposed by Cramer, this problem has been solved by Castillon in 1776.



We give here a complete solution, with proof based on ideas by Georges Lion.

- Let p, q, r be the inversions with centres P, Q, R, and preserving Γ ; $\Phi=p\circ q\circ r$ is an homography which preserves Γ, $\varphi=\Phi|_\Gamma$ its restriction on Γ.
- ABC solution $\Leftrightarrow$ B is fixed point for φ
- If $\varphi\neq Id|_\Gamma$, $\varphi(M)=M'$, $\varphi(N)=N'$; (MN’) and (M’N) intersect on a line Δ, the axis of φ.
- The fixed points of φ are the intersection points of Δ and Γ ; each of these points B defines a solution ABC : A=r(B) and C=p(B).
- If $\varphi=Id|_\Gamma$ then Φ is the inversion with Γ as set of fixed points, thus there are an infinity of solutions.

Appendices :

1. Menelaus’ theorem : an elementary proof.
2. Pascal’s hexagon theorem for a circle : a proof using Menelaus.
3. An interesting configuration : P, Q, R lie on a line which does not intersect Γ.
4. A nice property : (MM’) envelopes a conic.
5. Fixed points of an homography.

Maurice STARCK
Nouméa - New-Caledonia – 2002

WFNMC-4 Conference – Melbourne 2002

Castillon's problem

A circle G and three points P,Q and R (not on G) are given. Construct all triangles ABC inscribed in G and such as R Î (AB), P Î (BC) and Q Î (CA).

Proposed by Cramer, this problem, has been solved by Castillon in 1776. Pappus solved it in the particular case where P, Q and R lie on a line.
Solutions can be found in :

- "Exercices de Géométrie" by Frère Gabriel Marie
- "100 great problems of Elementary Mathematics" by Heinrich Dörrie
- "Géométrie" by Marcel Berger

It seems that the published solutions missed one outstanding case (§5).

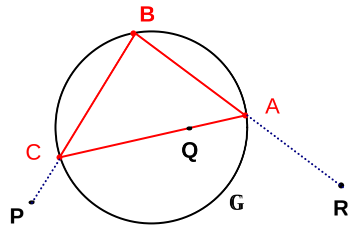
If the terms of the problem are elementary, and if the construction of all the solutions needs only a ruler, the working out of the method is not easy.
Here is a complete proof that uses anallagmatic geometry and ideas coming from results established by Georges Lion (University of New-Caledonia) ; the aim was to remain as elementary as possible, without resort to projective geometry.

1. **Let p, q, r be the inversions with centres P, Q, R, and preserving G ;**

$\Phi=p\circ q\circ r$ is a negative homography which preserves Γ, $\varphi=\Phi|_\Gamma$ its restriction on Γ.

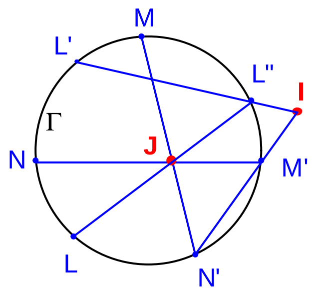
ABC solution $\Leftrightarrow$ B is fixed point for φ

$\varphi(B)=p\circ q\circ r(B)=p\circ q(A)=p(C)=B$



2. <u>lemma</u> : If $\varphi\neq Id|_\Gamma$ there exists two involutions i and j (inversions or reflexions) preserving Γ and such as $(i\circ j)|_\Gamma=\varphi=\Phi|_\Gamma$.

Let be M'=φ(M), N'=φ(N), L'=φ(L),
and J the intersection of (MN') and (M'N).
Necessarily we have $i\circ j$(M)=i(N')=M',
$i\circ j$(N)=i(M')=N', thus I lies on (M'N'),
and $i\circ j$(L)=i(L")=L', thus I lies on (L'L").
I is the intersection of (M'N') and (L'L").



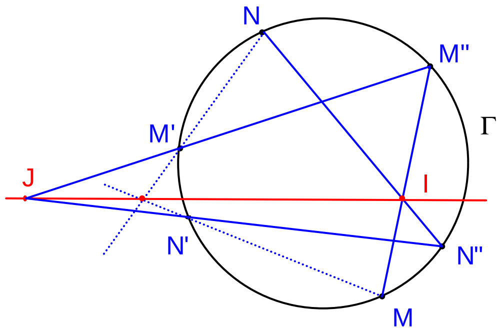
I and J being so introduced, they define i and j ; we have : $i\circ j$(M)=M'=φ(M), $i\circ j$(N)=N'=φ(N) and $i\circ j$(L)=L'=φ(L). Thus the negative homography $\varphi^{-1}\circ i\circ j$ has three fixed points on Γ and hence it fixes every point on Γ.
Remarks :

- If $\varphi=Id|_\Gamma$ (identity on Γ) then Φ is the inversion with Γ as set of fixed points ; this interesting case will be considered later in §5.
  If $\varphi\neq Id|_\Gamma$ then it has only two or less fixed points.
- If (MN')//(M'N) then J is at the infinity and φ is a rotation which has no fixed point.
- If (M'N')//(L'L") then I is at the infinity and i is a reflexion with axis a diameter of Γ.

To prove the existence of the **axis of j** we use Pascal's hexagon theorem (appendix 2).

If $M'=\varphi(M)$ and $N'=\varphi(N)$ then (MN') and (M'N) intersect on a fixed line Δ, the axis of φ.

$\varphi=(i\circ j)|_\Gamma$ $\varphi(M)=(i\circ j)(M)=i(M'')=M'$ $\varphi(N)=i(N'')=N'$
Let's consider the hexagon MM"M'NN"N' ;
(MM") intersects (NN") in I,
(M"M') intersects (N"N') in J,
hence (MN') intersects (M'N) on $(IJ)=\Delta$.

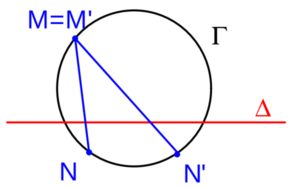


Remarks :

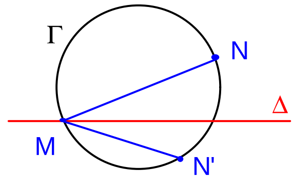
- Neither i nor j can't be the inversion with Γ as set of fixed points.
- If I is the centre of Γ then i is a negative inversion and $i|_\Gamma$ is the central symmetry.

3. The **fixed points of j** are the intersection points of Γ with the axis Δ of φ.

$\Rightarrow$ Let's suppose that $M\notin\Delta$ is a fixed point for φ, and let N be a non-fixed point.
Then the intersection of (MN') and (M'N) does not lie on Δ. Impossible.
A fixed point lies on Δ.

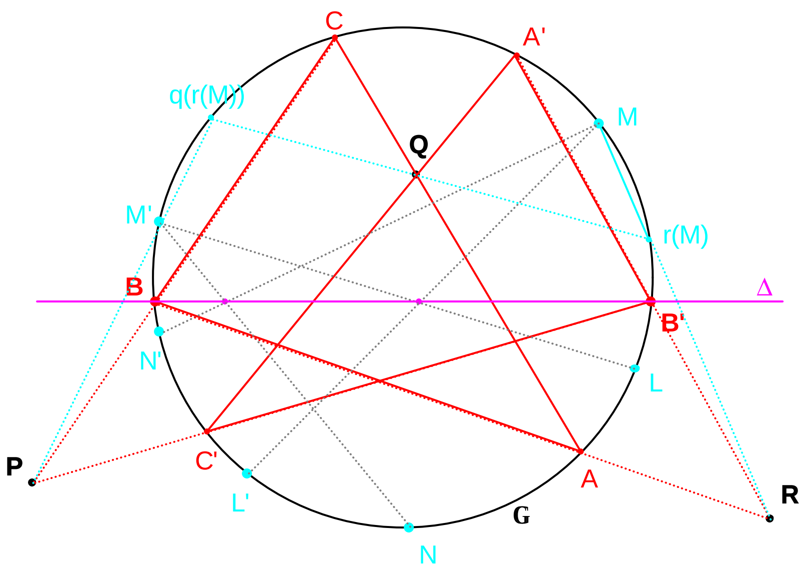


$\Leftarrow$ Let M be on Δ, and $N\neq M'$. (MN'), (M'N) and Δ are concurrent. (MN') intersects Δ in M, thus (M'N)=(MN) and M'=M.
A point on Δ is fixed.



4. **Construction of the triangles ABC** (Γ and P, Q, R are given)

- We choose three points M, N, L on Γ and we construct their images M', N', L' by φ.
- Two intersections, for example $(MN')\cap(M'N)$ and $(ML')\cap(M'L)$, define the axis Δ of φ.
- $\Delta\cap\Gamma$ is the set of fixed points B of φ : there are two (see example) or one or zero.
- If they exist we can now draw the triangles : (RB) and (PB) intersect Γ again in A and C.
  We verify that $Q\in(AC)$.



An interesting configuration is presented in appendix 3.

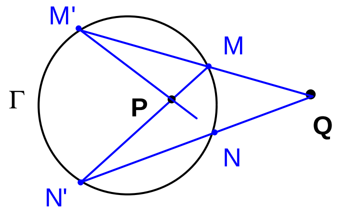
5. **An outstanding configuration** ($\varphi=\mathrm{Id}|_\Gamma \Leftrightarrow$ infinity of solutions)

<u>lemma</u> : p and q (inversions preserving Γ) commute

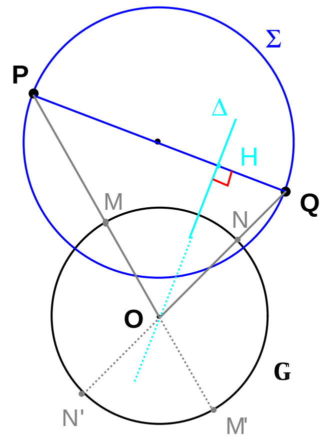
$\Leftrightarrow$ P lies on the axis of q $\Leftrightarrow$ Q lies on the axis of p

$\Leftrightarrow$ the circle with diameter [PQ] is orthogonal to Γ

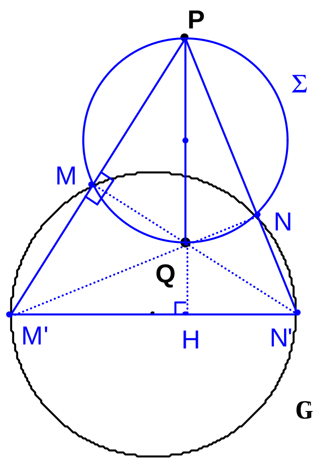
$\Leftrightarrow$ Let be M on Γ and $N=q\circ p(M)$,
and let's state $M'=q(M)$ and $N'=q(N)=p(M)$
We have : $p\circ q(M)=q\circ p(M) \Leftrightarrow p(M')=q(N') \Leftrightarrow p(M')=N$
$\Leftrightarrow P=(M'N)\cap(MN') \Leftrightarrow$ P lies on the axis of q
The same for $\Leftrightarrow$ Q lies on the axis of p.



$\Rightarrow$ p and q commute (and preserve Γ).
Let Σ be the circle with diameter [PQ], ρ the radius of Γ.
$p\circ q(Q)=p(\infty)=P$ and $q\circ p(P)=q(\infty)=Q$
$p\circ q(\Sigma)=q\circ p(\Sigma)=\Sigma$ since it is a circle or a line orthogonal to (PQ) and going through P and Q, thus $p(\Sigma)=q(\Sigma)=\Delta$ is a line orthogonal to (PQ) in $H=p(Q)=q(P)$.
$\overline{PH}\times\overline{PQ}=\overline{PM}\times\overline{PM'}=PO^2-\rho^2$ $\quad\overline{QH}\times\overline{QP}=\overline{QN}\times\overline{QN'}=QO^2-\rho^2$
(powers of P and Q with respect to Γ)
$(\overline{PH}+\overline{QH})\times\overline{PQ}=PO^2-QO^2$
$=(\overline{PH}+\overline{QH})\times(\overline{HQ}-\overline{HP})=(\overline{PH}+\overline{QH})\times(\overline{PH}-\overline{QH})=PH^2-QH^2$
$OP^2-OQ^2=HP^2-HQ^2$ thus $(OH)\perp(PQ)$ as level curve of Leibniz function $f(M)=MP^2-MQ^2$ and $(OH)=\Delta\perp\Gamma$
Finally $p(\Delta)=\Sigma\perp p(\Gamma)=\Gamma$ (p preserves orthogonality)



$\Leftarrow$ Let be $\Sigma\perp\Gamma$ in M and N, and $M'=p(M)$, $N'=p(N)$.
$(M'N')=p(\Sigma)\perp p(\Gamma)=\Gamma$ hence (M'N') is a diameter of Γ
and $(PQ)\perp(M'N')$ in $H=p(Q)$.
$Q\in(MN')$ since $(MP)\perp(MQ)$ and $(MM')\perp(MN')$
hence $N'=q(M)$, just as $M'=q(N)$.
$p\circ q(M')=p(N)=N'$ $\;p\circ q(N')=p(M)=M'$ $\;p\circ q(\infty)=p(Q)=H$
$q\circ p(M')=q(M)=N'$ $\;q\circ p(N')=q(N)=M'$ $\;q\circ p(\infty)=q(P)=H$
$p\circ q$ and $q\circ p$ are two positive homographies which coincide in three points, hence they are equal :
$(p\circ q)\circ(q\circ p)^{-1}=(p\circ q)\circ(p\circ q)=\mathrm{Id}$ since it is positive and has three fixed points.



<u>theorem</u> : $p\circ q\circ r$ is the inversion γ which set of fixed points is Γ

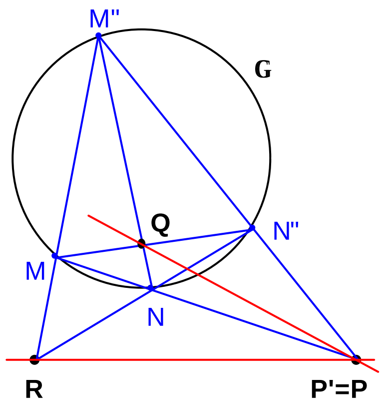
$\Leftrightarrow$ p, q and r commute two by two

$\Leftrightarrow$ the circles with diameters [PQ], [QR] and [RP] are orthogonal to Γ

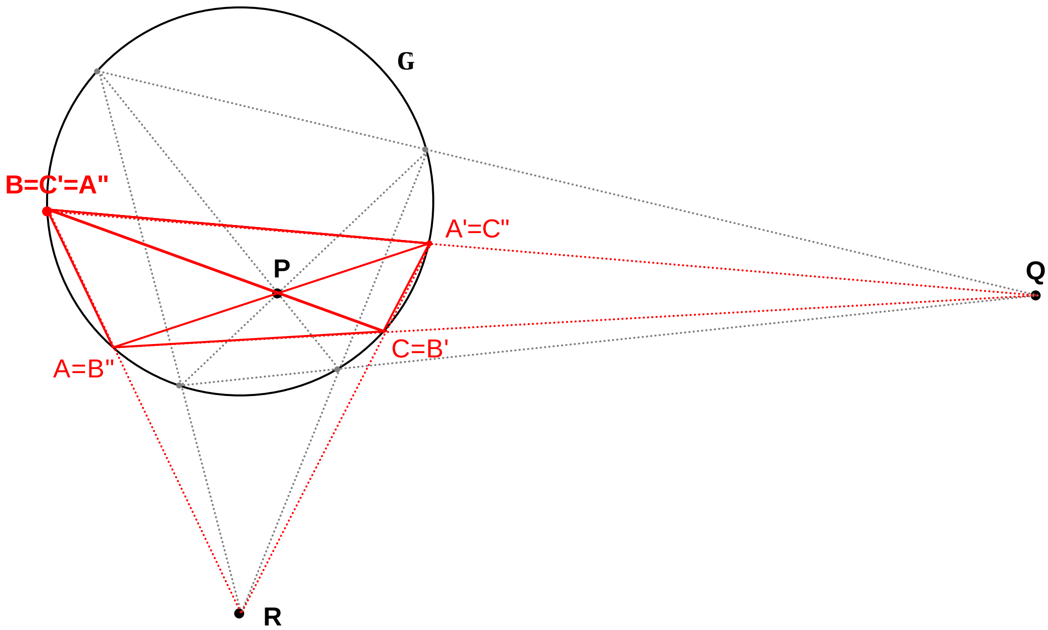
The second $\Leftrightarrow$ follows from the first thanks to the lemma. Let's prove the first.

$\Rightarrow$ $p\circ q\circ r=\gamma$ hence p, q and r commutes with γ (as inversions which preserve Γ)
$p\circ q=\gamma\circ r=r\circ\gamma=(\gamma\circ r)^{-1}=(p\circ q)^{-1}=q\circ p$, just as $q\circ r=r\circ q$
$\gamma=(p\circ q)\circ r=(q\circ p)\circ r=q\circ(p\circ r)$ thus $p\circ r=q\circ\gamma$ and we have also $p\circ r=r\circ p$

⇐ p, q and r commute two by two, hence P is the intersection of the axis of q and r (lemma)
N=q∘r(M) N"=q(M)
r(N")=N and r(M)=M", thus P'=(MN)∩(M"N") lies on the axis of r.
Likewise q(M)=N" and q(M")=N thus P' lies on the axis of q.
Hence P'=P, and for every M on Γ we have p∘q∘r(M)=M.



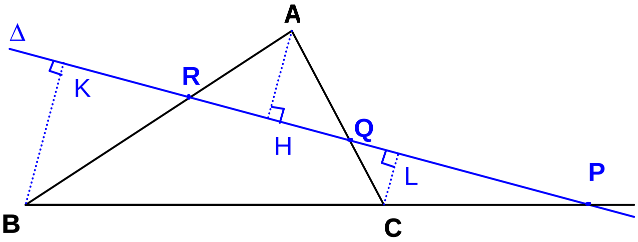
A simple way to draw this configuration : chose four points on the circle Γ and draw the complete quadrangle. The three intersection points are P, Q and R (in any order). Any point on Γ provides three required triangles ABC (since all three vertices are fixed points, the chosen point may be any of them).



appendix 1 : **Menelaus' theorem**

> ABC is a triangle, R∈(AB), P∈(BC) and Q∈(CA).
>
> P, Q, R on a line Δ $\Leftrightarrow$ $\dfrac{\overline{PB}}{\overline{PC}}\times\dfrac{\overline{QC}}{\overline{QA}}\times\dfrac{\overline{RA}}{\overline{RB}}=+1$

$\Rightarrow$ Let H, K and L be the orthogonal projections of A, B and C on Δ.



First we consider just distances :

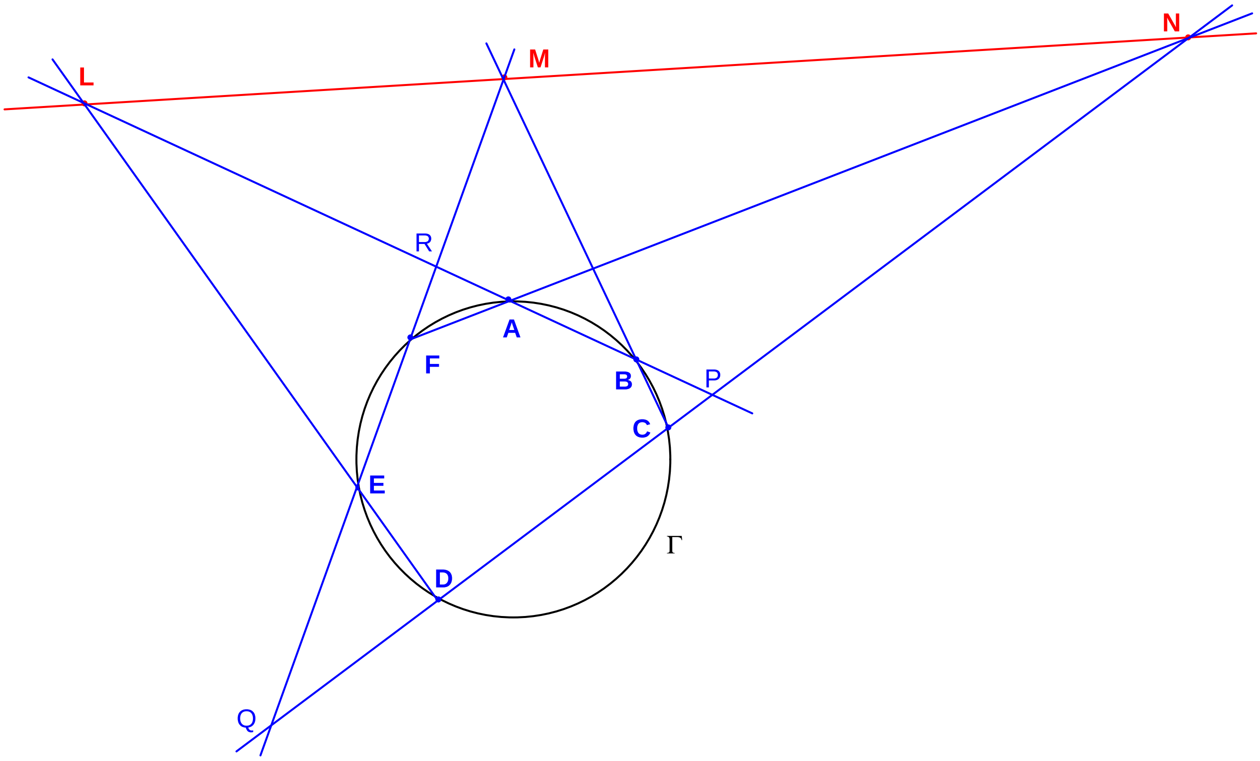
$$\frac{PB}{PC}=\frac{KB}{LC} \quad \frac{QC}{QA}=\frac{LC}{HA} \quad \frac{RA}{RB}=\frac{HA}{KB} \quad \text{which product is 1}$$

If ABC lies on one side of Δ then the three ratio are positive, hence the product is +1.
If Δ intersects ABC then two ratio are negative and the product is still +1.

$\Leftarrow$ $\dfrac{\overline{PB}}{\overline{PC}}\times\dfrac{\overline{QC}}{\overline{QA}}\times\dfrac{\overline{RA}}{\overline{RB}}=+1$ and $\dfrac{\overline{PB}}{\overline{PC}}\neq+1$ thus $\dfrac{\overline{RA}}{\overline{RB}}\neq\dfrac{\overline{QA}}{\overline{QC}}$ hence (QR) and (BC) are not parallel.
Let then P' be the intersection of (QR) and (BC) ; now using $\Rightarrow$ with the line (P'QR) we get $\dfrac{\overline{P'B}}{\overline{P'C}}=\dfrac{\overline{PB}}{\overline{PC}}$ and hence P=P'∈(QR).

appendix 2 : **Pascal's** theorem (for a circle)

If ABCDEF is an hexagon inscribed in a circle then the three intersection points (AB)$\cap$(DE), (BC)$\cap$(EF) and (CD)$\cap$(FA) of the pairs of opposite sides lie on a line.



Using Menelaus' theorem in the triangle PQR with the lines (DE), (BC) and (FA) we get :

$$\frac{\overline{LR}}{\overline{LP}}\times\frac{\overline{EQ}}{\overline{ER}}\times\frac{\overline{DP}}{\overline{DQ}}=+1 \qquad \frac{\overline{MQ}}{\overline{MR}}\times\frac{\overline{BR}}{\overline{BP}}\times\frac{\overline{CP}}{\overline{CQ}}=+1 \qquad \frac{\overline{NP}}{\overline{NQ}}\times\frac{\overline{FQ}}{\overline{FR}}\times\frac{\overline{AR}}{\overline{AP}}=+1 \qquad (1)$$

Using the powers of P, Q and R with respect to Γ we get :

$$\overline{PC}\times\overline{PD}=\overline{PB}\times\overline{PA} \qquad \overline{QD}\times\overline{QC}=\overline{QE}\times\overline{QF} \qquad \overline{RA}\times\overline{RB}=\overline{RF}\times\overline{RE} \qquad (2)$$

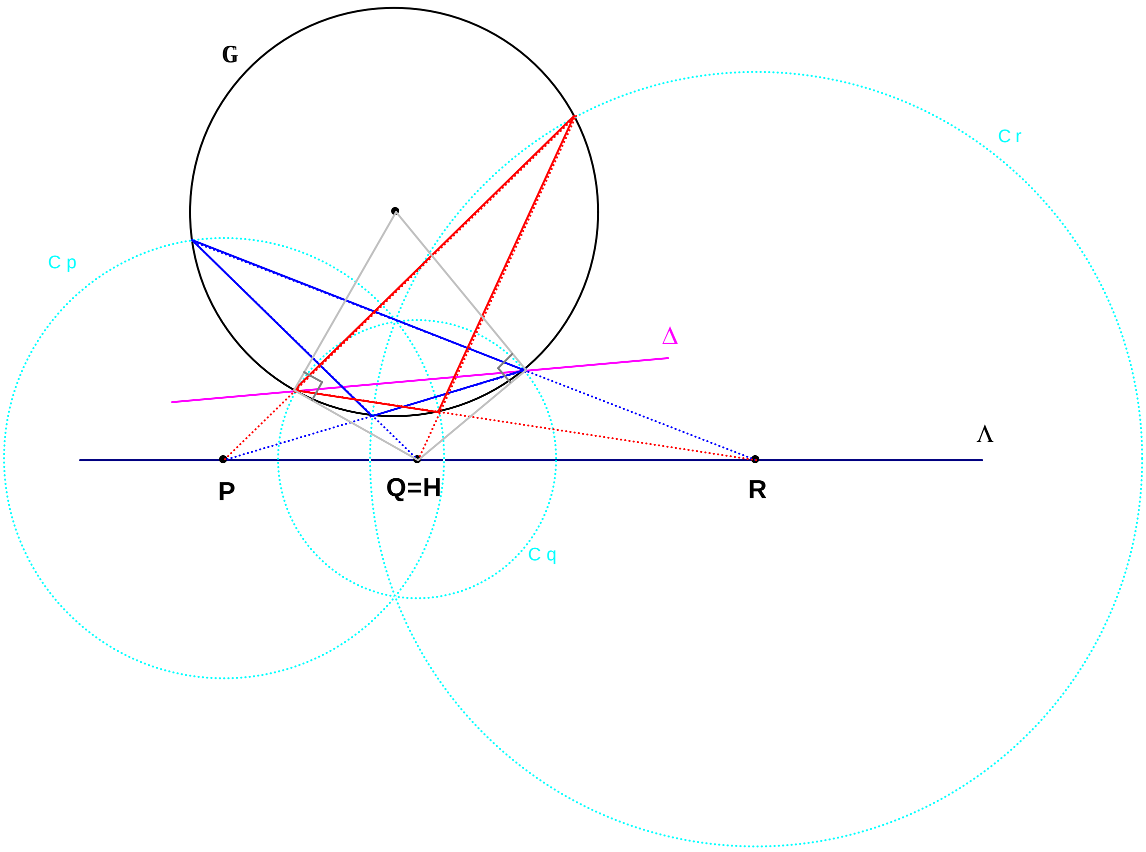
Now the product of the relations (1), taking account of the equalities (2) gives

$\frac{\overline{LR}}{\overline{LP}}\times\frac{\overline{MQ}}{\overline{MR}}\times\frac{\overline{NP}}{\overline{NQ}}=+1$ hence (Menelaus) L, M and N lie on a line.

<u>appendix 3</u> : **An interesting configuration** : P, Q, R lie on a line Λ which doesn't intersect Γ.

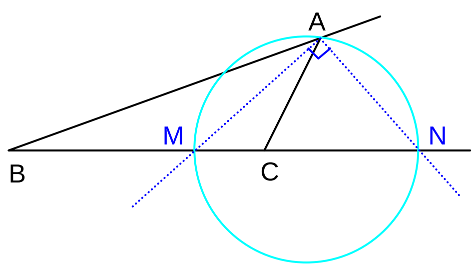
In this configuration Φ=p∘q∘r is an inversion with positive power preserving Γ whose pole H lies on Λ, and the tangents to Γ starting from H define the two fixed points.
The three inversion circles C_p, C_q and C_r of p, q and r are concurrent in two points symmetric with respect to Λ (one of them is inside Γ).

Moreover if these three circles intersect each other with angles of 60°, then they are the **Apollonius' circles** of the two triangles (and H=Q, hence Φ=q).



Reminder : **Apollonius' circles of a triangle ABC**.

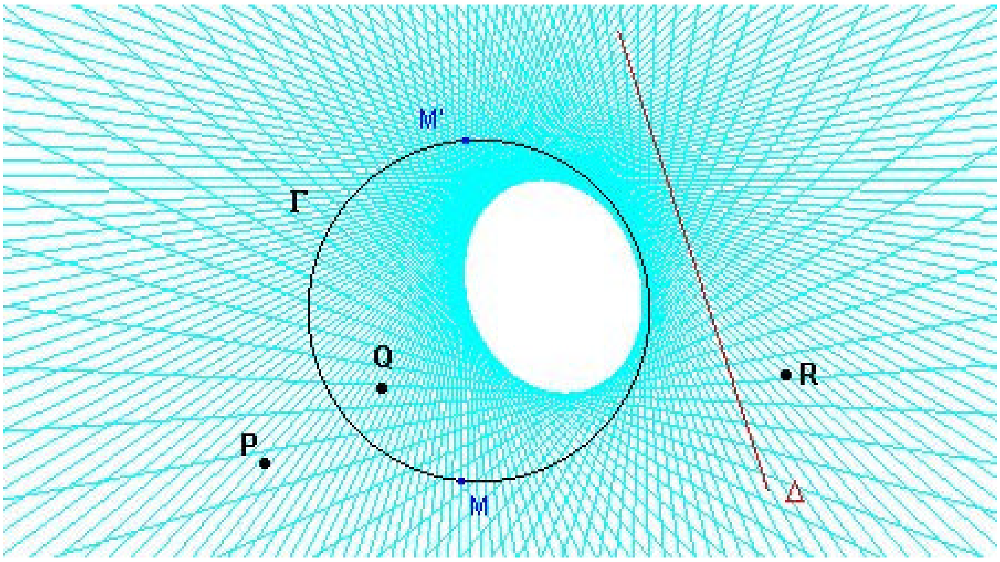
The Apollonius' circle with respect to the vertex A is the circle with diameter [MN], where M and N are the intersection points of the two angle bisectors starting at A with (BC).
The three Apollonius' circles of ABC intersect each other in two points with angles of 60°.

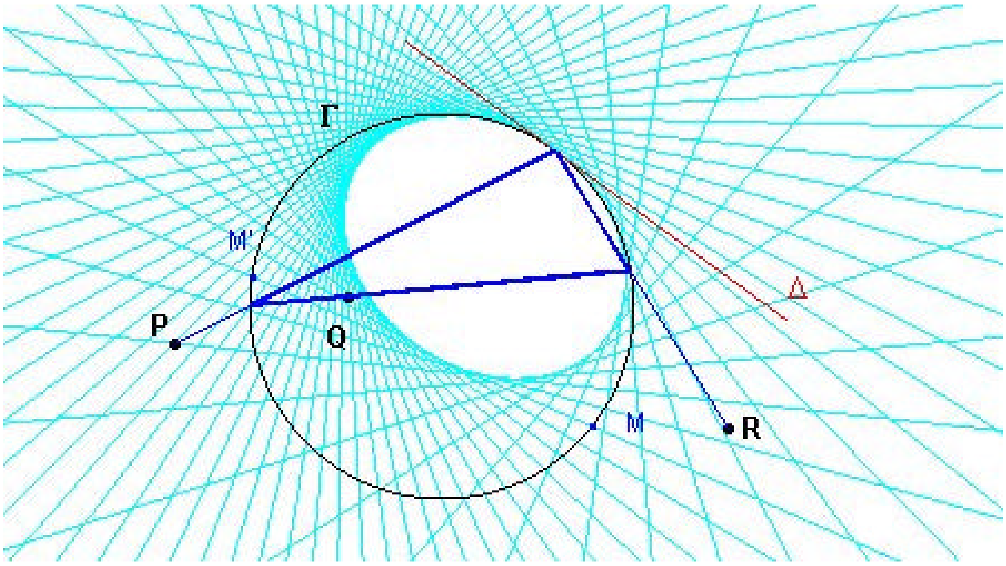


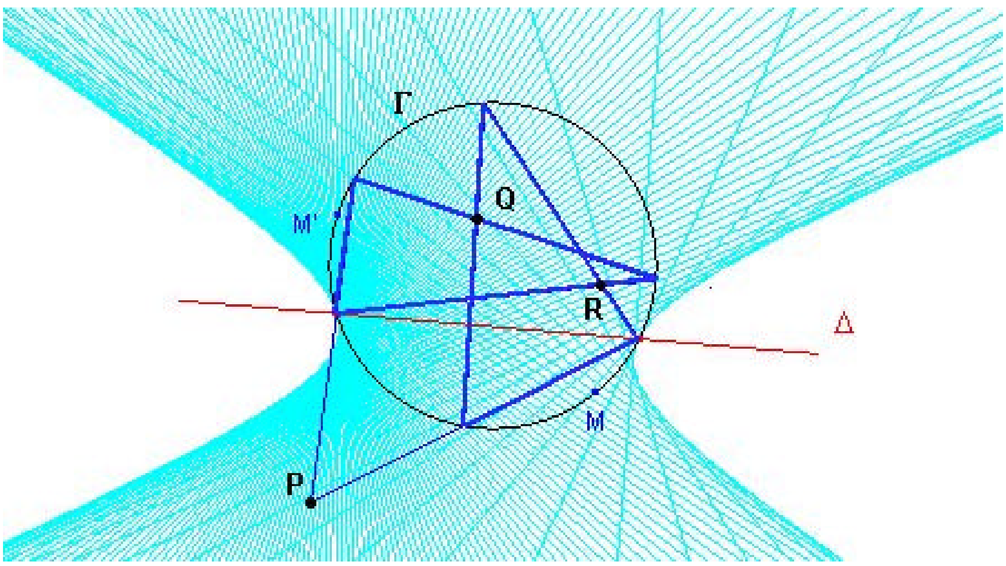
Remark : If Λ intersects Γ and the three points P, Q, R, or only one of them, lie inside Γ then the Castillon's problem has no solution.

appendix 4 : **A nice property** (Georges Lion, under publication)

The line (MM') envelopes a conic ; if this conic is (bi)tangent to Γ then the fixed point(s) are the contact point(s). In the configuration considered in §5 the conic is Γ.







appendix 5 : **Fixed points of an homography**

$Z=\dfrac{az+b}{cz+d}$ or $Z'=\dfrac{a\bar{z}+b}{c\bar{z}+d}$ where a, b, c, d are complex and $ad-bc\neq 0$ (Z and Z' non constant)

If $c\neq 0$ we can assume that c=1 and thus use a, b, d as $\frac{a}{c}$, $\frac{b}{c}$, $\frac{d}{c}$.

- $Z=z \Leftrightarrow z^2+(d-a)z-b=0$ hence we get $\boxed{\text{two or one fixed points}}$
- $Z'=z \Leftrightarrow z\bar{z}+dz-a\bar{z}-b=0 \Leftrightarrow \left| \begin{array}{l} z\bar{z}+\mathcal{Re}(dz-a\bar{z}-b)=0 \\ \mathcal{Im}(dz-a\bar{z}-b)=0 \end{array} \right.$

  $\Leftrightarrow \left| \begin{array}{l} \text{circle (second degree with } z\bar{z}=x^2+y^2\text{) or } \varnothing \\ \text{the plane or a line (first degree) or } \varnothing \end{array} \right.$

  hence we get $\boxed{\text{a circle of fixed points (inversion), or two or one or zero fixed points}}$

If c=0 then we get :

- equation of first degree, hence one (direct similitude) or zero (translation) fixed point
- two equations of first degree, hence the intersection of two lines :
  a line of fixed points (reflexion), or one (inverse similitude) or zero (glide) fixed point