68.34 A polyhedron and its volume

G. Haigh ended his discussion of glass rods (Note **67.29** October 1983) with the polyhedron obtained by removing from a unit cube all the 24 tetrahedra similar to that indicated in Fig. 1, and he asked for both an illustration and a calculation of its volume.

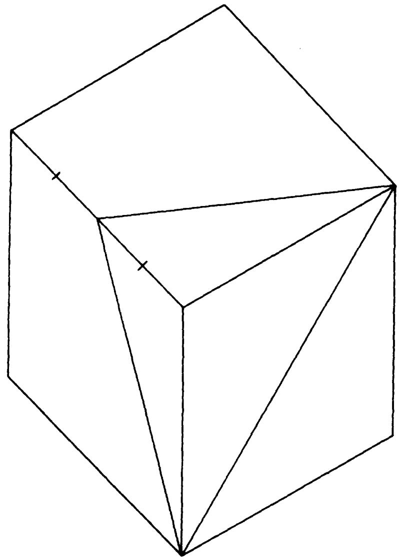

FIGURE 1.

The resulting polyhedron is illustrated in Fig. 2. Note that each cutting plane determines one of the 24 congruent kite faces. Hence each edge is a segment of the line of intersection of two cutting planes and so each vertex

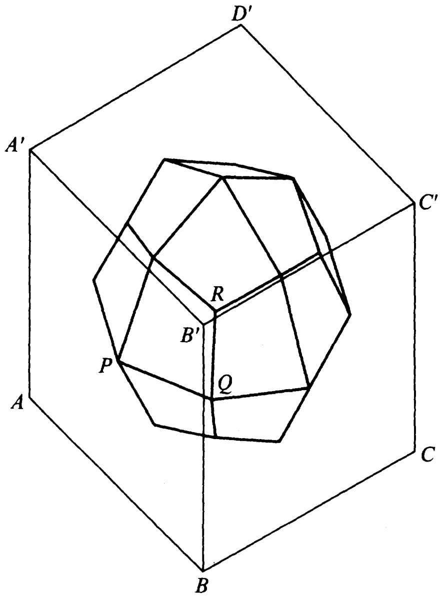

FIGURE 2.

lies on such a line. Also, the edges and vertices lie on the planes and lines of symmetry of the cube. (It may also be of interest that the polyhedron has three cross-sections, parallel to faces of the cube, which are regular octagons, and four cross-sections, perpendicular to diagonals of the cube, which are regular hexagons.)

Referring to Fig. 2, it follows that P is the centre of the square $ABB'A'$. To determine the positions of Q and R: let X be the point on the top face of the cube, determined as in Fig. 3(a). Then Q and R are as shown in the cross-section $B'D'DB$ of Fig. 3(b).

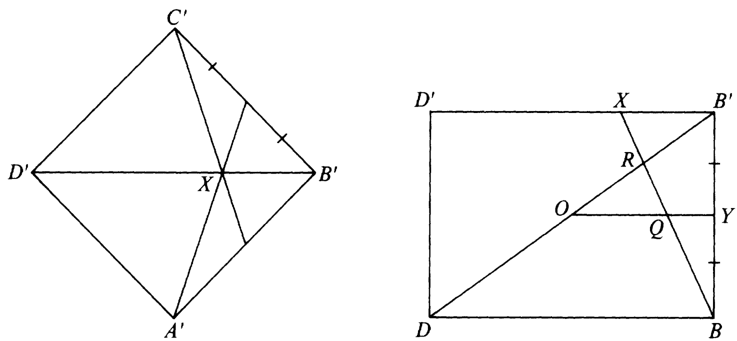

FIGURE 3.

If O is the centre of the cube we have, by similar triangles,

$$\frac{B'X}{XD'} = \frac{1}{2} \quad \text{and thus} \quad \frac{B'R}{RO} = \frac{1}{1} \quad \text{and} \quad \frac{OQ}{OY} = \frac{2}{3}.$$

Taking O as origin, and x-, y-, z-axes parallel to CB, AB, AA', we therefore have

$$P = (\tfrac{1}{2}, 0\ 0),\ Q = (\tfrac{1}{3}, \tfrac{1}{3}, 0) \quad \text{and} \quad R = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}),$$

$$\therefore \text{ the volume of } OPQR = \tfrac{1}{6}\begin{vmatrix} \tfrac{1}{2} & 0 & 0 \\ \tfrac{1}{3} & \tfrac{1}{3} & 0 \\ \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{4} \end{vmatrix} = \tfrac{1}{144}.$$

Hence the total volume of the polyhedron $= 48 \times \frac{1}{144} = \frac{1}{3}$.

A. K. JOBBINGS

Bradford Grammar School, Bradford BD9 4JP