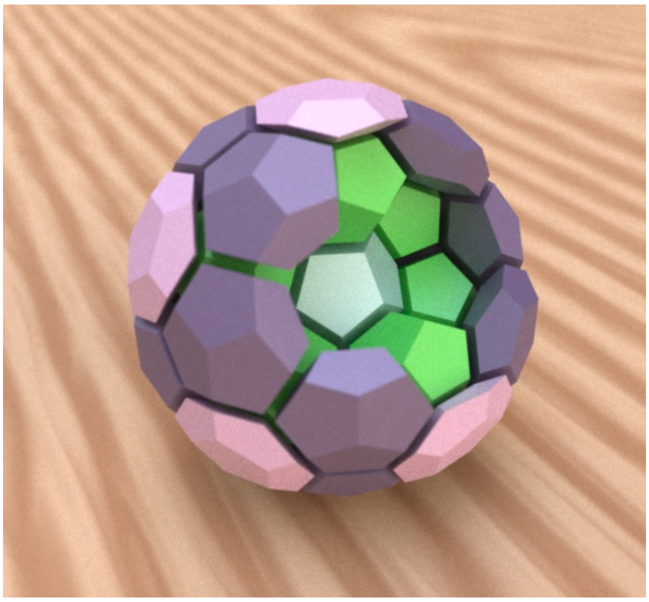
The shadow of the 120

A 3D puzzle from the fourth dimension



Computer generated image. We removed a few pieces to see the inside.

Table of contents

Activity

The puzzle is almost self-sufficient.

It is relatively easy. The colours help. People take between 2mn and 20mn to put it back together. Some children are very quick. There is something pleasant in reconstructing the object. It is relatively easy to guess which pieces get where but their orientation is a bit disconcerting.

The minimal indications should be to point out the cup and that one has to put the pieces back there so that the construction can hold together. If there is a hat, explain that it is there to close the object. In the version where 2 pieces are glued to a polygon, these have to be put last, on top.

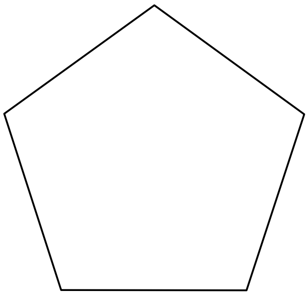
More hints would include the existence of a core and a colour-coded onion-like structure. Also, the pieces get more flat as they get far from the core. Try identifying which pieces go behind the windows. The placement of the first two pieces help a lot: I recommend using violet ones.

Is is much easier if the player sees the assembled object and watches it being taken apart.

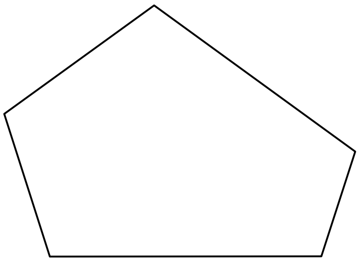
After it has been solved, the facilitator can reveal that the geometry comes from the fourth dimension and answer any question that the visitor can have.

Mathematical explanations

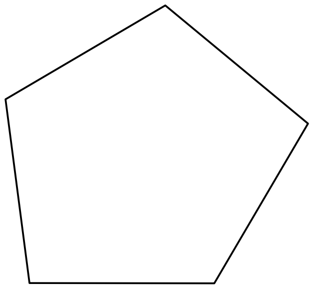
A *regular polygon* is a polygon that is as symmetric as possible. So we want all sides to have the same length and all angles at vertices to be the same.



Regular

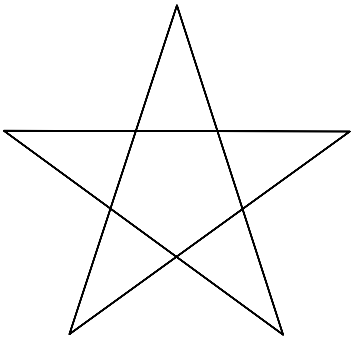


Angles: OK
Lengths: not identical



Angles: not identical
Lengths: OK

We exclude self-crossing polygons.



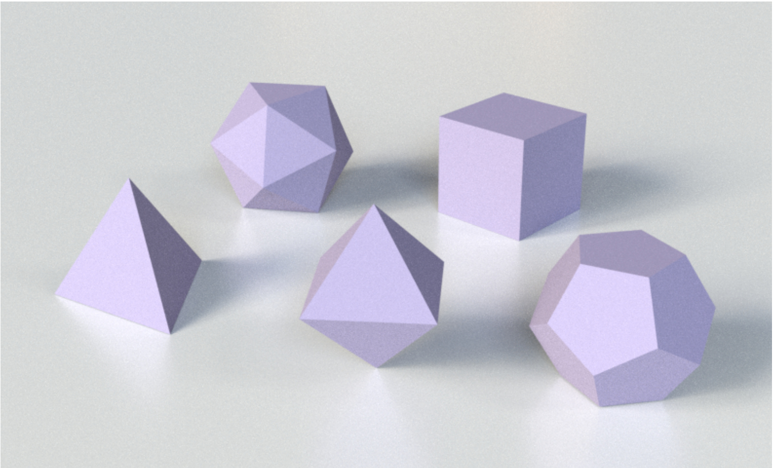
A regular polyhedron is a polyhedron that is as symmetric as possible. We also exclude self-intersection. All faces must be regular and identical. Moreover, all vertices must be shared by the same number of faces. These conditions turn out to be sufficient. Then we get more: the group of symmetries (isometries, preserving orientation or not) of the shape is *transitive on flags*, which means that if one chooses any two faces, and a side of each, and a vertex on each side, then there is a symmetry of the object that simultaneously matches the chosen faces, sides and vertices. In particular, the angle between the faces are equal, the shape looks the same at each vertices, but more.

There are only 5 regular polyhedra, called the *platonic solids*.

These are:

- the regular tetrahedron: 4 regular triangles, 3 per vertex
- the cube: 6 squares, 3 per vertex

- the regular octahedron: 8 regular triangles, 4 per vertex
- the regular dodecahedron: 12 regular pentagons, 3 per vertex
- the regular icosahedron: 20 regular triangles, 5 per vertex



Now 4D regular polytopes: their facets are identical regular polyhedra, arranged in a way as symmetric as possible. The official definition has already been given above: the symmetry group shall be transitive on flags. However, there is here too a simpler criterion, once we have taken identical regular facets: just ask the organization near each edge to be identical and the folding angle to be the same everywhere.

Now there are 6 possibilities:

- the 5-cells, a.k.a. pentachoron or 4-simplex, facets=reg. tetrahedra, 3 per edge
- the 8-cells, a.k.a. octachoron or tesseract or hypercube, facets=cubes, 3 per edge
- the 16-cells, a.k.a. hexadecachoron, facets=reg. tetrahedra, 4 per edge
- the 24-cells, a.k.a. icosatetrachoron or octaplex, facets=reg. octahedra, 3 per edge
- the 120-cells, a.k.a. hecatonicosachoron, facets=reg. dodecahedra, 3 per edge
- the 600-cells, a.k.a. hexacosichoron, facets=reg. tetrahedra, 5 per edge

This list coincides with the following procedure: take a certain number of copies of one regular polyhedron. Try to place them, side against side, around one edge. You must fit at least 3 of them.

Q: How many can you fit ?

A: 3 or 4 or 5 regular tetrahedra fit around an edge. Or 3 octahedra, or 3 cubes, or 3 dodecahedra.

In higher dimension, the list continues but surprisingly has only 3 elements for each dimension n: a generalized tetrahedron: the n-simplex, a generalized cube, and a generalized octahedron.

This makes the 120-cells one of very few exceptional gems.

Counting

The 120-cells has 600 vertices, 1200 edges, 720 faces of dimension 2 (pentagons), and of course 120 facets.

Its symmetry group has $120\times12\times5\times2=14400$ elements, this is 120 times more than for the regular dodecahedron.

At each edge meet 3 facets, at each vertex meet 4 facets.

Measuring

The angle between facets is 144°, i.e. 2/5 of a full turn, which is remarkable because an irrational angle would have been expected (for instance, the angle between faces for the icosahedron has a cosine of $-1/\sqrt{5}$ and is not a rational number of full turns). In other words, the folding angle is 1/10 of a turn. This turns out to have an explanation: there are closed chains of 10 facets touching along faces and whose centres are aligned in a common plane passing through the centre of the 120-cell.

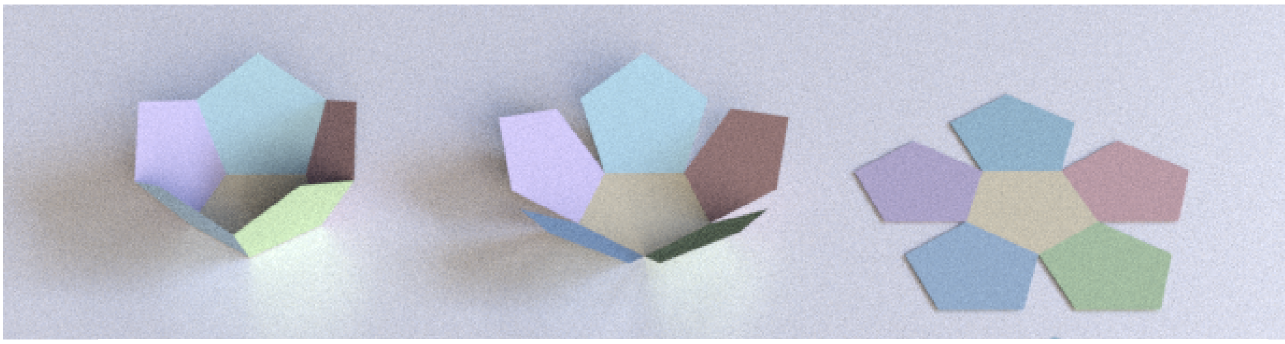
All the vertices belong to a hypersphere (set of points at a given distance from a given point).

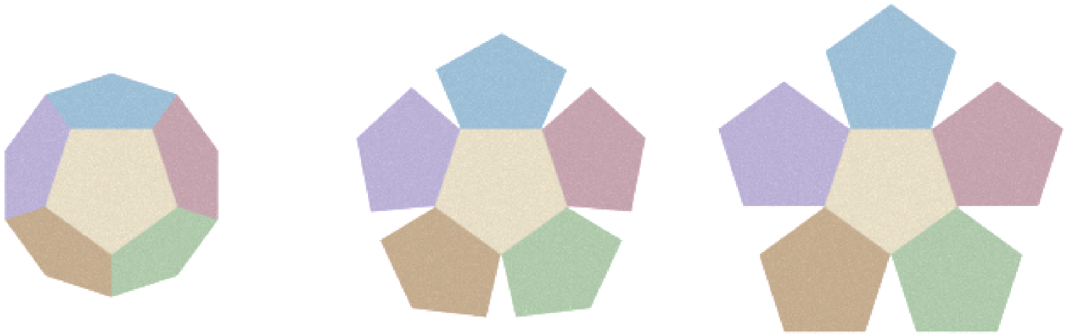
If we fix an edge length of 1, then the smallest disk containing the pentagon has radius ≈ 0.851 (diameter ≈ 1.701), the sphere containing the dodecahedron a radius ≈ 1.401 (diameter ≈ 2.803) the hypersphere containing the whole object a radius ≈ 3.702 (diameter ≈ 7.405).

Many more properties

On Wikipedia!

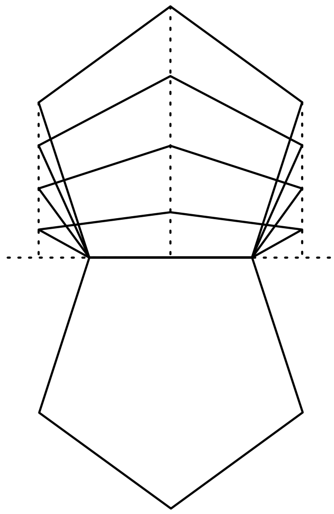
Understanding the mathematical model

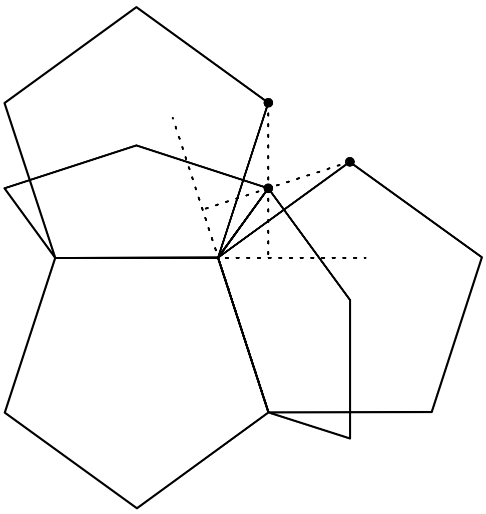




Take a (regular) dodecahedron and place it on a table. The contact zone is a pentagon. It touches 5 other pentagons. Together, they make the bottom half of the dodecahedron. Under an orthogonal projection this half hides the upper half. This is illustrated on the left part of the picture above.

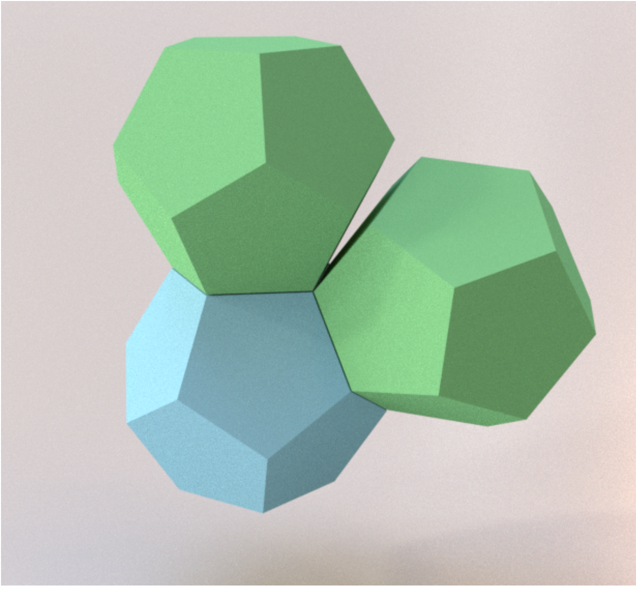
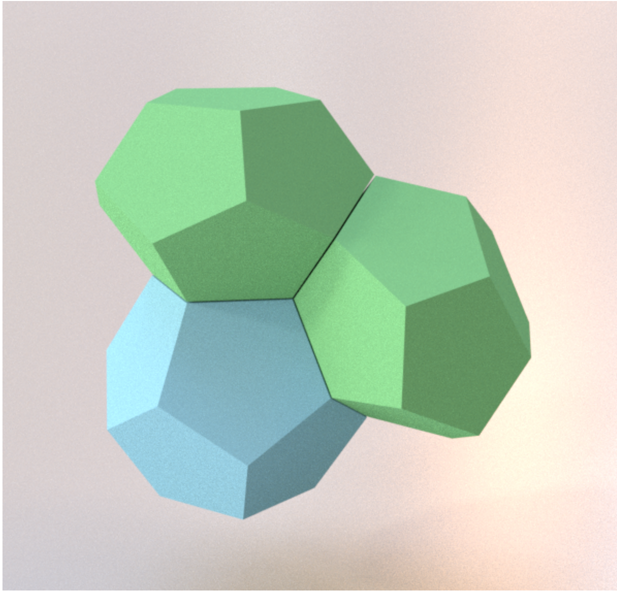
Unfold the bottom half. This gives a flat object, 6 regular pentagons, 5 around a central one. See the right part of the picture.





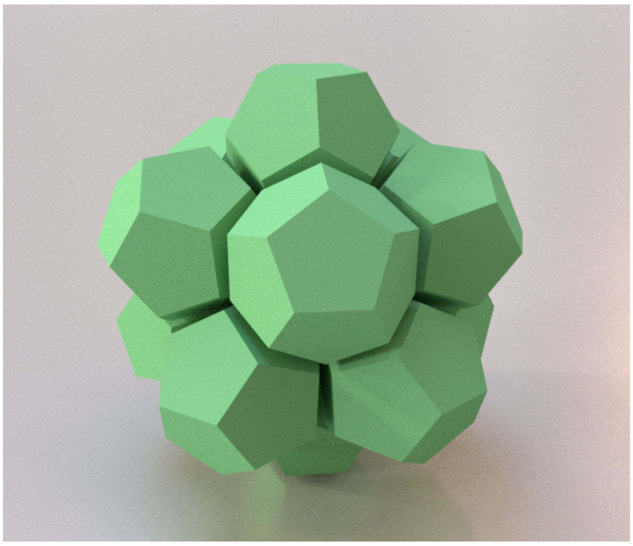
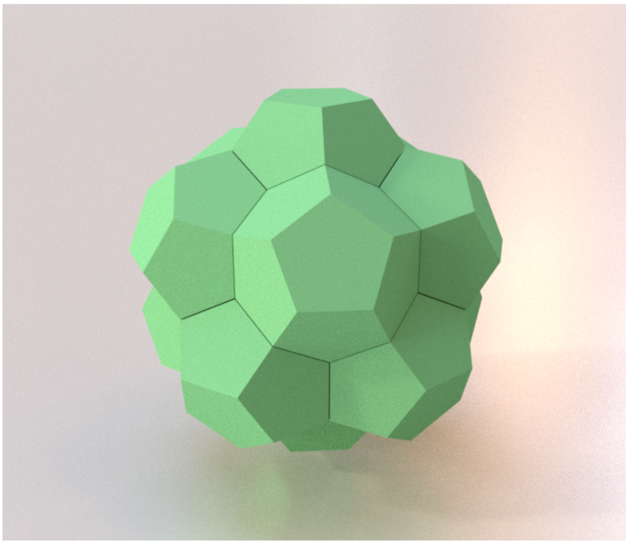
Imagine it being folded again and think of the projection. Each face is rotated around one edge, and the effect on the projection is to shorten it along the direction in the plane that is orthogonal to the edge, while preserving size in the direction along the edge, see the figure on the right. The shortening factor is the cosine of the folding angle: f=cos(a).

With a little bit of classical and/or Cartesian geometry, we can determine the factor f, whence the angle a, see the figure on the left.

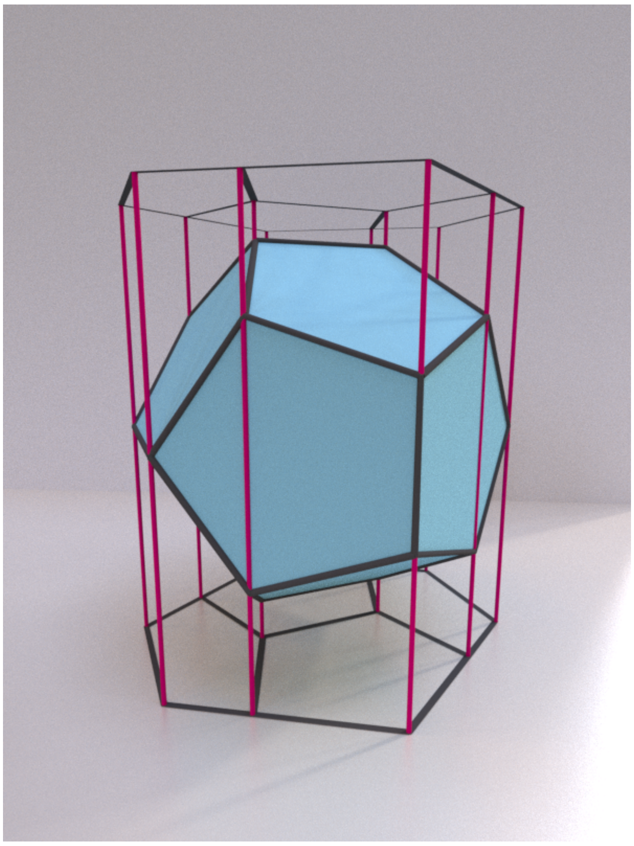
Similarly, the 120-cells is oriented in 4D-space so that the bottom facet is parallel to the 3D-space on which we project. Hence as for the dodecahedron, this facet is projected without deformation: this is the central piece of the puzzle, cyan blue in the image above. If we try to put 3 dodecahedra, touching along their pentagons, around an edge, then there remains a small angular gap.

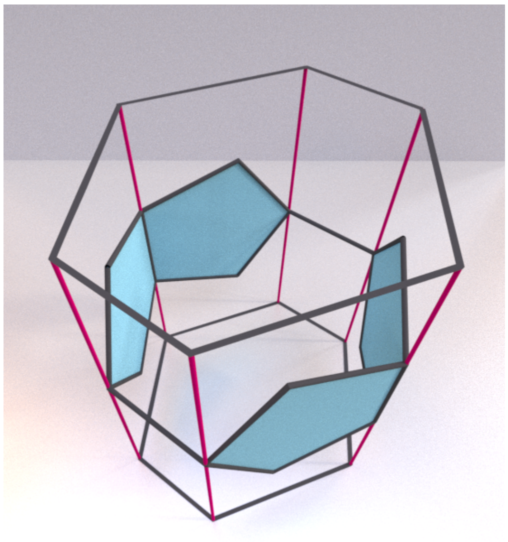
The bottom facet of the 120-cell touches 12 other facets, let them be green, that make a slight angle with the 3-space, hence their projections are slightly flattened in the direction orthogonal to the pentagon of contact with the blue. We can place in 3D-space 12 copies of the blue icosahedron touching its 12 faces and "fold" them in the 4D space, and we see their projection that shrink along the 12 normals to the 12 blue pentagons, until the small gap is reduced to nothing.

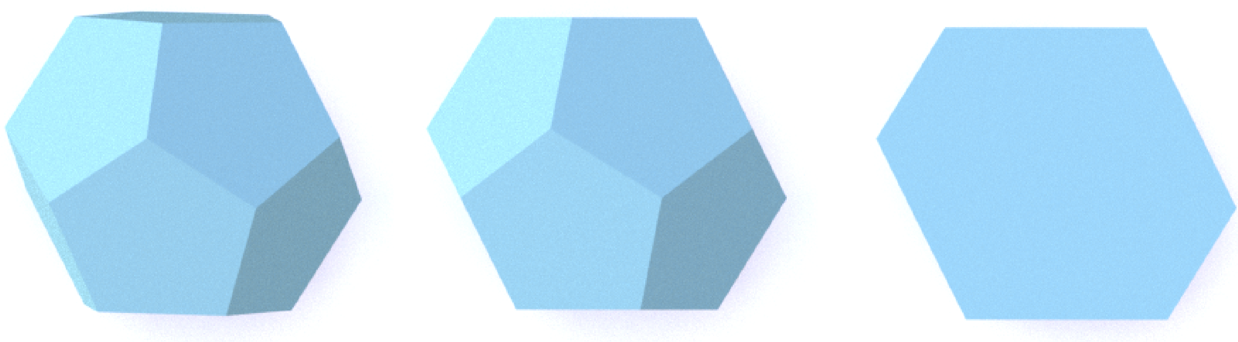
Let us now think of the cube. Take a square in a plane and place 4 other squares touching the 4 edges. Now fold it in 3D and think of the projection to the plane. The folding angle is 90°. The projection of a folded face is a segment! These four segments form a shell that completely encloses the square.

An interesting example is the regular dodecahedron balancing on one edge. On the projection (see the image on the left) we only see 4 polygons. It turns out that there are 4 polygons projected to lines and 4 situated on the top. The top and bottom polygons are aligned in pairs and projected down to the same polygon. There is a difference with the case of the cube above: if you take the faces that are projected to lines, those lines do not cover the whole contour of the projected figure. This is illustrated on the right.

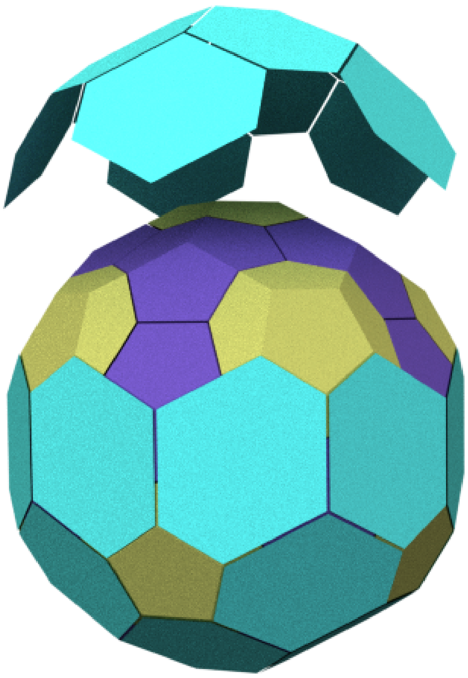




Below in the centre figure we show what the orthogonal projection photograph would look like. On the left we rotated the object slightly before the projection. On the right we removed the inner details to show only the outline.

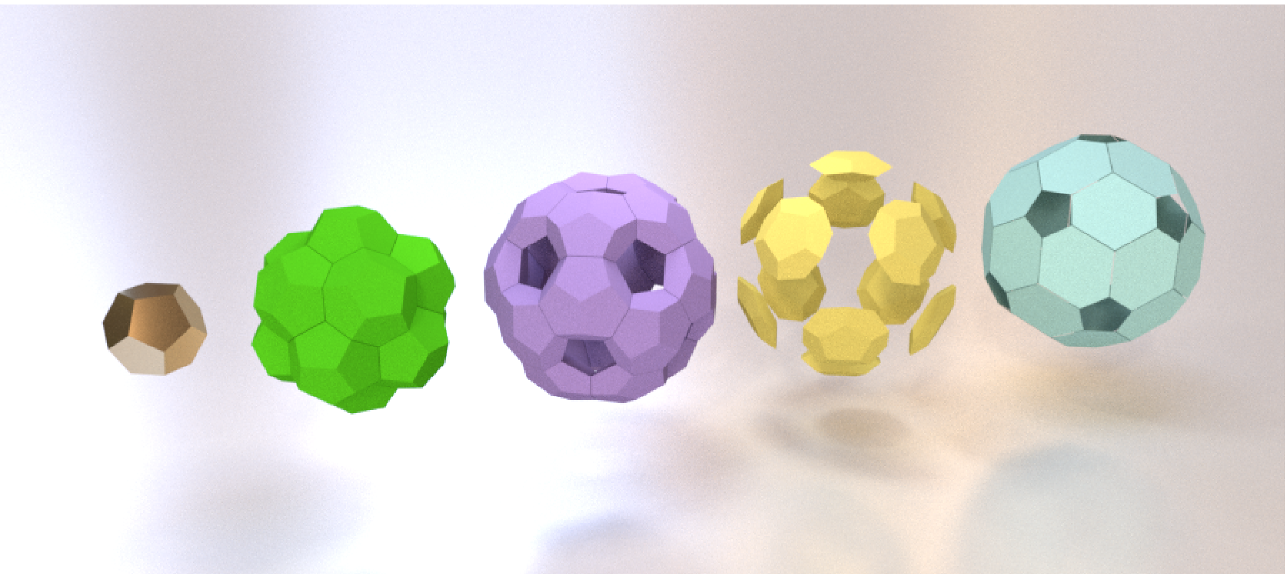


Two similar thing are happening for the projection of the 120-cell to 3D-space:

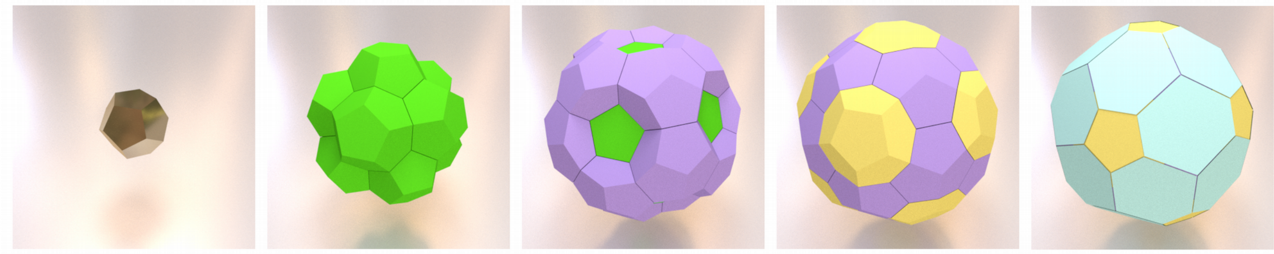


- 90 of the 120 dodecahedra come in pairs projected to the same piece.
- The remaining 30 are projected flat because they make an angle of 90º with the 3D-space. Together, they make a shell enclosing the whole projection, but this shell has holes, where one can see (yellow) pentagonal faces of some of the inner non-flat pieces. These pieces touch their symmetric piece exactly on this pentagon.

The Pieces of the puzzle come in 4 non-flat flavours, plus the flat version used in the shell. This is because the angle of the dodecahedra with the 3D space, thus flattening ratio of the piece, depend on the distance of the centre of the piece to the centre of the puzzle. They are also not all flattened according to the same axis.

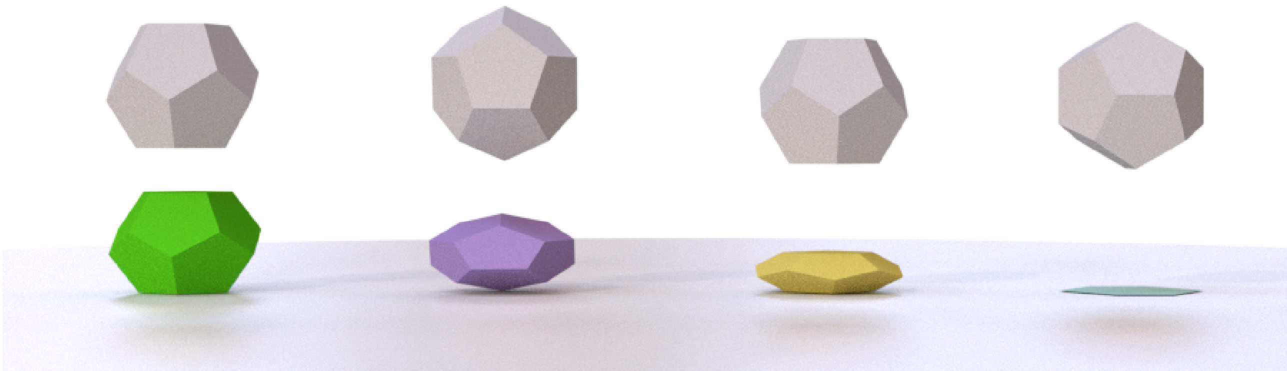


The successive shells of the puzzle around the central piece. Each has respectively 1, 12, 20, 12 and 30 pieces.

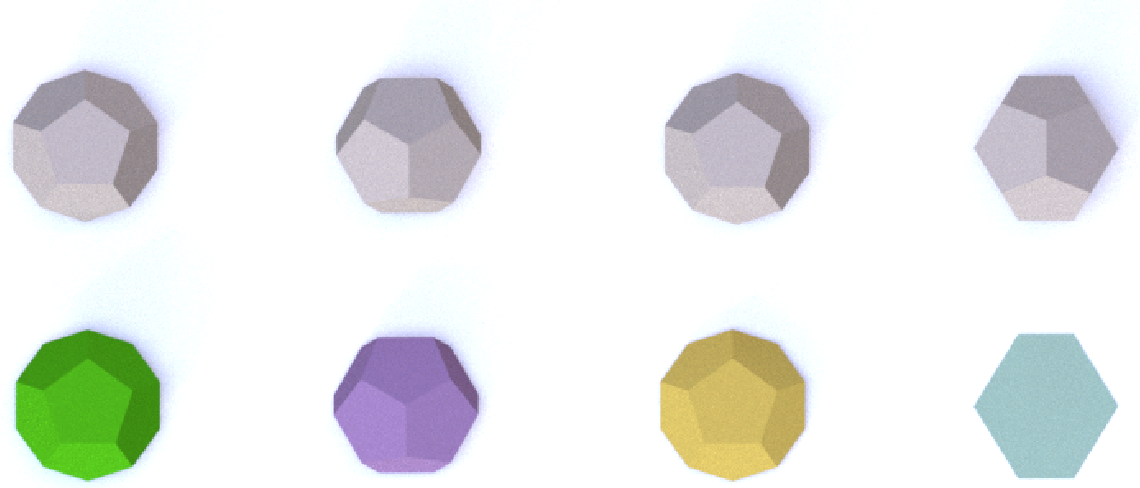


Reconstructing the projected object layer by layer

Below: dodecahedra, before and after flattening.

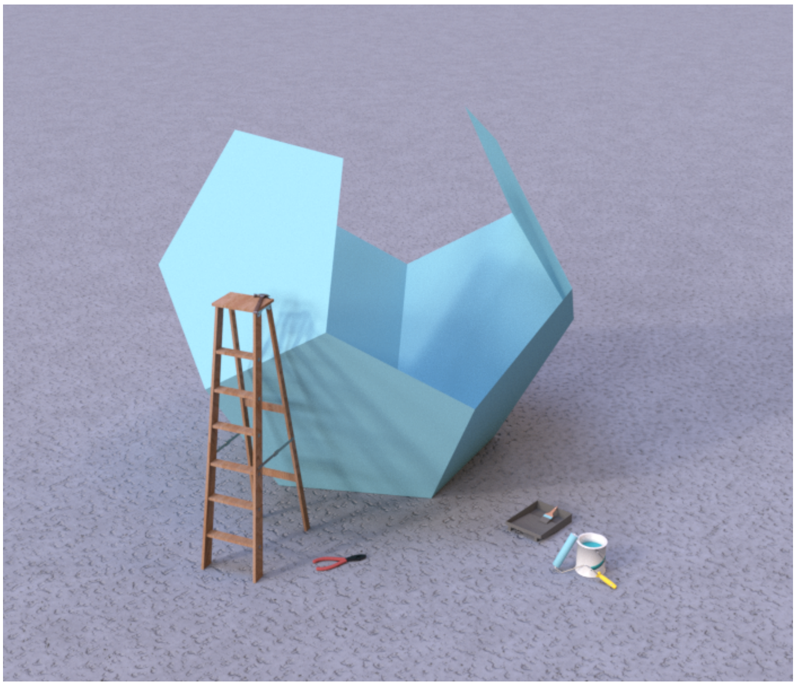


Perspective view from the side



Orthographic view from above

Constructing the mathematical model



If you ask a mathematician what it means to construct the 120-cell, s/he will probably reply that it is "*proving the existence of a polytope made of 120 regular dodecahedra and on which the symmetry group is transitive on flags*". This is—not—what we will do here.

The puzzle pieces

If we are only interested in the shape of the 5 types of piece in the puzzle, all we need is:

- a model of the dodecahedron,
- for each type, the axis of the flattening
- and the amount of flattening.

We have already seen which axes the flattening must have, and we know the flattening factor for three out of 5:

- central piece: no flattening
- layer 1: axis perpendicular to a face, factor $f_1 = \cos(144°)$
- layer 2: axis through a pair of opposite vertices, factor f_2 = ?
- layer 3 : axis as in layer 1, factor f_3 = ?
- layer 4 : axis through midpoints of opposite edges, factor $f_4 = 0$ (flat!)

As it turns out, the factors are quite simple:

- $f_1 = \phi/2 = 0.809017\ldots$
- $f_2 = 1/2 = 0.5$
- $f_3 = \phi - 1 = 0.618034\ldots$

where $\phi = (1 + \sqrt{5})/2$ is the golden mean.

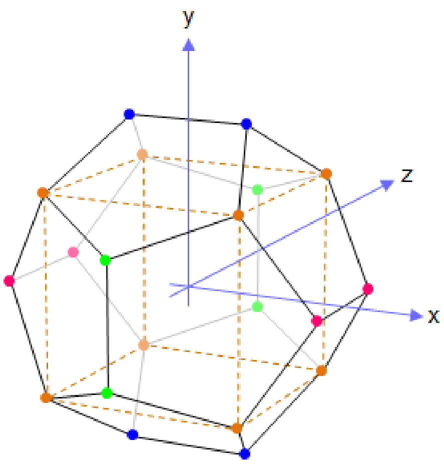
Vertices of the regular dodecahedron

Mathematical models of the regular dodecahedron can be found on the Internet. Of course it depends on what we mean by modelling and there are many possibilities. We may for instance want a list of vertex coordinates (and a way to group them into faces). Or a method to define the planes containing the faces. Or a set of instructions to define points (ruler and compass style). Or an abstract definition by a set of characteristic properties. Etc.

We keep our practical application into mind and give a particular set of vertex coordinates, shamelessly borrowing it from Wikipedia: take all the points of the form

$$(\pm1, \pm1, \pm1)$$
$$(0, \pm1/\phi, \pm\phi)$$
$$(\pm1/\phi, \pm\phi, 0)$$
$$(\pm\phi, 0, \pm1/\phi)$$

where the ± are independent. This defines the 20 vertices.



The figure on the right has a coordinate system which is left-handed with y-axis up. I am used to right-handed with z-axis up, I'll try to avoid mistakes.

Here the side length $(2\times r_1) = 2/\phi$. You have to scale it if you want another value for r_1.

Here also, a segment mid-point is on the vertical axis.

If you want on this axis a face centre instead, you have to rotate the object along the horizontal axis labelled x on the figure by an angle that can be determined by some classical geometry: it is the angle $A=\widehat{C_2C_3C_1}$ where C_1 is the segment midpoint on top, C_2 is the centre of a face touching the segment, and C_3 is the centre of the dodecahedron. This triangle is rectangle at C_2. There are several ways to get the value of the angle, it is for instance half of the folding angle: A = 37.7175…°. Another way is to use the explicit coordinates and work out the coordinates of the three points, then the cosine of A by a vector product.

If you want a vertex on the vertical axis, you have to rotate along the horizontal axis labelled z on the figure, by an angle $A=\widehat{C_0C_3C_1}$ where C_0 is one of the endpoints of the segment on top. Here also the triangle is rectangle, this time at C_1. One gets A = 20.9052…°.

How do you perform these rotations? See the toolbox near the end of this document. Note that you do not really need the value of the angle: what you actually need is its sine and cosine.

The 120-cell

Now one may be interested in how the dodecahedra are placed in 4D-space to form the 120-cell. This may be useful for instance if one wants to create a variant of the puzzle where the 120-cell is oriented differently in 4D space before being projected to 3D space. We may also be interested in measuring the object, and in the methods that can be used to discover some of those values.

So let us consider the 120-cell. Each facet is a regular dodecahedron. The regularity condition on the 120-cell implies in particular that all facet centres lie at the same distance from a common point, i.e. they are on a hypersphere.

Centre it on the origin, in a chosen coordinate system (x,y,z,t). We want this system to be orthonormal, so that vectors (x,y,z,t) have length $\sqrt{x^2+y^2+z^2+t^2}$.

Now if we know the ratio between the radius r_4 of the hypersphere and the size of the dodecahedron, we are basically done, as explained later.

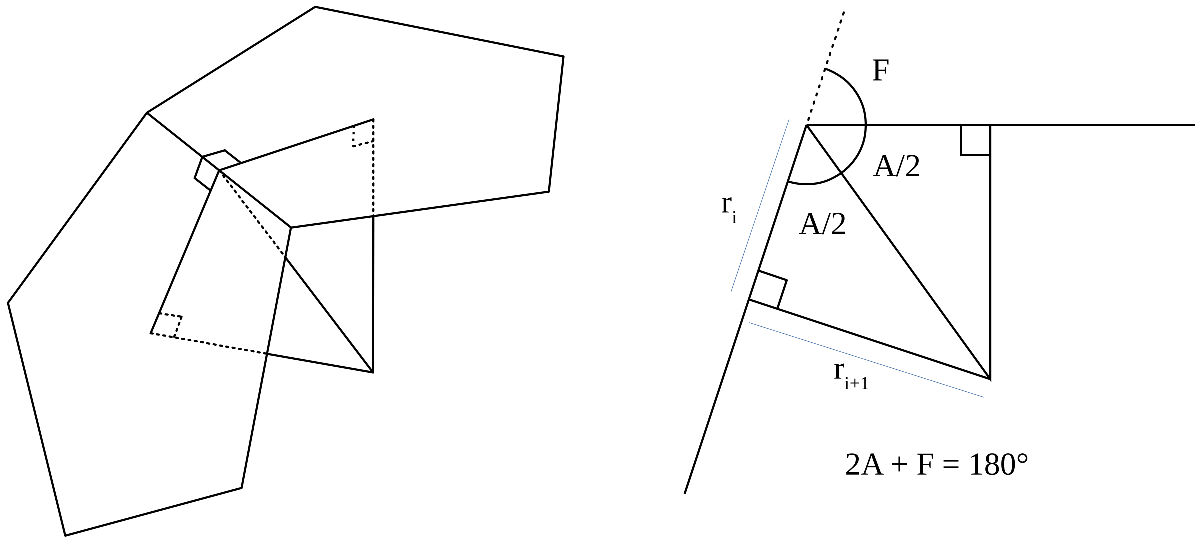
There are several size measures that can be attached to a dodecahedron: the edge length, the radius of the sphere containing its vertices, or the midpoints of all segments, all face centres, etc.… We choose to use the radius of the sphere containing the face centres and call it r_3.

Consider also the distance from a face (pentagon) centre to a segment mid-point, call it r_2, and call r_1 the half length of a segment. Then:

$$\frac{r_4}{r_3}=\tan\frac{A_3}{2} \qquad \frac{r_3}{r_2}=\tan\frac{A_2}{2} \qquad \frac{r_2}{r_1}=\tan\frac{A_1}{2}$$

Where A_1, A_2, A_3 are the angle between the consecutive segments/faces/facets.

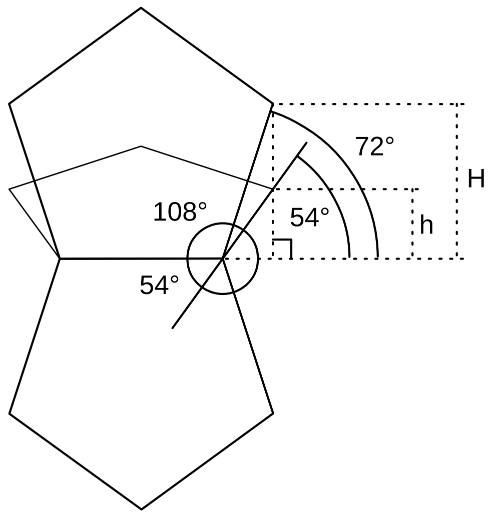
This is illustrated below.



Left: two faces and the dodecahedron. Right: section by the plane passing by the triangles. Proportions and angles modified on the right for more readability.

Often, it is easier to get first the folding angle F and deduce A from it, because 2A + F = 180°.

For instance, $5F_1 = 360°$ because we have a regular pentagon and must distribute one full turn to 5 corners, whence $A_1/2 = 54°$, $\tan(A_1/2) = 1.376381…°$



The angle F_2 has a cosine equal to the quotient h/H in the figure besides, so $\cos F_1 = \tan 54° / \tan 72°$ whence $A_2/2 = 58.2825…°$ $\tan(A_2/2) = 1.618034…°$ In fact it turns out to be the golden mean.

The angle F_3 may be determined similarly and with a bit of work one finds $F_3 = 36°$, which is remarkable. Whence $A_3/2 = 72°$, $\tan(A_3/2) = 3.077686…°$

Summary:

- $r_2/r_1 = \tan(A_1/2) = 1.376381…$
- $r_3/r_2 = \tan(A_2/2) = 1.618034…$
- $r_4/r_3 = \tan(A_3/2) = 3.077686…$

Once we have r_1, …, r_4 the distance between any pair of the following 5 points in the 120-cell can be determined easily:

- centre C_4 of the 4D object,
- centre C_3 of a facet (dodecahedron),
- centre C_2 of a 2D face (pentagon) of the facet,
- midpoint C_1 of an edge of this face
- and endpoint C_0 of this segment.

That is because the vectors whose length we measured by

$$r_1=d(C_0,C_1),\ r_2=d(C_1,C_2),\ r_3=d(C_2,C_3) \text{ and } r_4=d(C_3,C_4)$$

are all orthogonal. So for instance the radius of the sphere passing through all vertices is $d(C_0,C_4) = r_1^2+r_2^2+r_3^2+r_4^2$.

Reflections

We choose a coordinate system (x,y,z,t) for 4D-space and (x,y,z,t) → (x,y,z) for the projection to 3D-space. We centre the 120-cell on the origin.

If we want one dodecahedral facet to be projected without any flattening then its supporting plane has to be parallel to the (x,y,z,0) hyperplane. It has to be on top or at the bottom and the distance from this plane to the centre of the 120-cell has to be r_4.

In other words we take all the vertex coordinates in 3D space of our regular dodecahedron (with a size determined as in the study above), and define for each such triple (x,y,z) a point in 4D-space (x,y,z,r_4). This gives our first, topmost, facet. Its centre is $(0,0,0,r_4)$.

A neat way to deduce the position of other facets in the 120-cells is via *reflections*. Indeed, consider a facet f, and a (pentagonal) face thereof. Consider then the reflection across the hyperplane passing by the centre C_4 and containing the face. The image of the facet f by this reflection is precisely the facet that touches f on the chosen face. This is the exact analogue of what happens for regular polyhedra, subtracting one dimension to all considered objects.

Take our first facet, the one that we placed on top. The 12 reflections associated to the 12 faces give us 12 facets that project to the first layer (in green in our illustrations) around the central piece of the puzzle (in cyan blue or gold in our illustration). The next layer (violet) has 20 facets and since they all touch green facets by faces, they can be deduced from them by appropriate reflections. One can work out a set of 20 appropriate green faces by closely looking at the pictures. The next layer of t=12 facets (yellow or pink) is obtained by reflecting each green ones using its face opposite to the central piece. The outer shell is made of 30 (blue) facets, they all touch 2 violet and 3 yellow/pink ones, any of which can thus be used to construct them. They somehow are in correspondence with the edges of the central dodecahedron of the puzzle. Their centres are all in the hyperplane t=0 and they all get projected flat in 3D-space.

As a variant, one can compute one polyhedron of each colour, and then use (part of) the symmetry group of the dodecahedron to construct the rest. Each symmetry is here to be understood as acting on (x,y,z,t) by leaving t unchanged and modifying (x,y,z).

Another variant is to use the flattening factor to deduce the angle between the dodecahedra centres as viewed from the 120-cell centre, and knowing that the projected centres will be aligned to specific direction with respect to the central projected dodecahedron, one can also work out 4D coordinates.

Whatever is the chosen method, the rest of the 120-cell can be deduced from the 45 first facets (excluding the 30 blue ones), by a central symmetry with respect to the origin of the 4D-space: $(x,y,z,t) \rightarrow (-x,-y,-z,-t)$.

Here is another variant for the lazy one: explore the tree of all possible reflections, keeping a list of facet centres, and cutting a branch every time it grows to a facet whose centre is already in the list. With a computer, that uses floating point number representations, one can compare centres by setting some small d under which they are considered as equal. This distance must be small compared to $2 \times r_3$ but big enough to take rounding errors into account. Brutally this would require computing 12×119 = 1428 reflections and a number of comparisons of the order of 7000. There are a few possible optimizations (though with today's average computer speed, this shall not be necessary).

Constructing a computer model of the 120-cell and its projection

Representations

We have to agree on what we want to have in the computer's memory in the end.

An elegant possibility would be to build a *linear cell complex*. We'll aim at something simpler but it is interesting to be aware of this possibility. This is:

- An indexed list of vertices given by 4D coordinates (each vertex is 4 floats).
- An indexed list of edges, each being a couple of vertex indices. Edges won't be repeated.
- An indexed list of faces, each being a collection of 5 edge indices (preferably circularly ordered).
- An indexed list of facets, each being a collection of 12 face indices.

In fact it is useful too to have orientation information too, which is slightly more complicated: edges and faces can be randomly oriented, then to each cell of dimension d, the orientation implies an orientation of its bounding cells of dimension d-1, which we compare to the chosen one by a boolean (true/false) indicating it it is the same or the opposite. Edges can be implicitly oriented by the order of its vertices.

Here we will aim at a different representation that is slightly less general and less linked, and has some amount of redundancy :

- A list of 120 dodecahedra, each being an indexed list of 12 vertices given by their coordinates, together with a boolean for each dodecahedron, related to orientation considerations.

We do not merge identical vertices, so the same vertex will appear 4 times because each vertex belongs to 4 dodecahedra.

The vertices are numbered from 1 to 12 (or 0 to 11) and each dodecahedron is an image of the top one by an isometry of 4D-space such that the numbering of vertices match. The boolean then indicates whether this isometry is orientation preserving or reversing.

The faces are then given by the same data for the 120 dodecahedra: a length 12 list whose elements are lists of 5 indices. This means that after an isometry applied to one dodecahedron, the new dodecahedron inherits a numbering of its faces and vertices.

Building

The dodecahedra are built by layers as explained in the previous section, using reflection across faces (we saw that there are other methods, but here we only explain this one). The 120-cell is centred on the origin. The first dodecahedron is placed at $t=r_3$. To perform the reflections one needs a function that takes the coordinates of the face vertices (3 vertices are enough: they cannot be aligned and together with the centre of the 120-cell they define the 3D-plane of reflection that is all we need) and a list of (20) points given by their coordinates and returns a list of (20) reflected points. And don't forget to invert the orientation flag.

What I did is to first do Schmidt orthonormalization* on the triple of vectors from the origin to the three chosen vertices in the pentagonal face (*: look that up on the Internet or in a book). Now a reflection with respect to the hyperplane passing through the origin and directed by an orthonormal basis a,b,c is simply

$$v \to 2p - v$$

where p is the projection of v on the hyperplane, which can be computed as

$$p = (v \cdot a)a + (v \cdot b)b + (v \cdot c)c.$$

The list of reflections that have to be applied can be deduced by careful observation of the successive layers. It amounts to a list of 119 elements of the form [F,f] where F is the number of a previously constructed facet, so strictly less than the index of [F,f] in the list, and f is a face number between 1 and 12 (or 0 and 11). I reduced it to 59 elements by using the central symmetry of the 120-cell, $v \to -v$. It may be even reduced even further.

Determining this list may be partially automated. For instance it begins with (I use the usual programming convention where indices begin with 0):

[0,0], [0,1], [0,2], [0,3], … , [0,11].

There are many possibilities that will give the right result, and I leave it to the reader to make one's choice.

Projecting

If one wants to reconstruct the puzzle presented here, just apply the projection

$$(x,y,z,t) \to (x,y,z)$$

to each vertex. You only need to do it for the 45 dodecahedra above the equator, i.e. whose centre has a positive t (and maybe also the 30 of the shell, whose centres have t=0).

One may want to correct the orientation of the dodecahedra whose orientation are marked as reversed: it depends on the 3D software that will use the data but usually one must give the vertices in a face in some particular order, according to the outward pointing orientation. Recall our vertices are designated by their indices between 1 and 20 (or 0 and 19) and faces by a list of 5 indices. Just reverse the order of this length 5 list for each face.

If you ever insist on projecting a facet whose centre was below the equator (i.e. t<0) then its orientation flag has to be reversed, I think.

Rotating in 4D

Another thing you might want to do is to use a different projection. My preference go to the orthogonal projections on a different 3D space. This is equivalent to first “rotate” the model in 4D-space (by rotate I mean in fact perform an isometry) and then project it by $(x,y,z,t) \to (x,y,z)$.

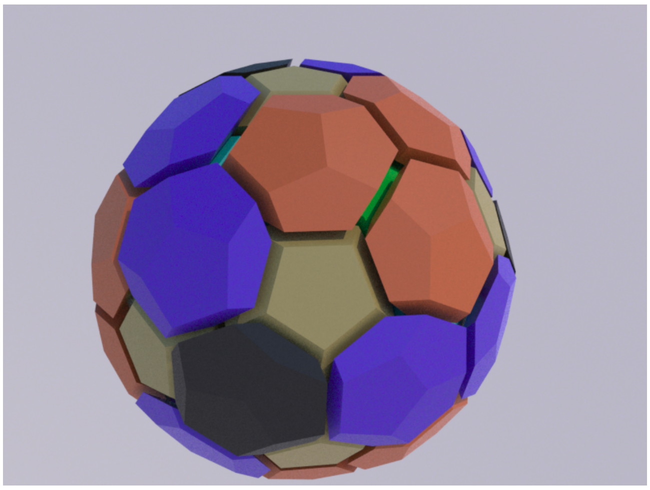
See the toolbox near the end of this document for more information on performing rotations.

Examples of projected objects

A few examples are pictured on the next page. By having a given point “on top” we mean that before the projection the 120-cell undergoes a isometry fixing the origin so as to place the given point so as to maximize its t coordinate, which is equivalent to having the point on the t-axis (all other coordinates are 0) with t>0.

We use the term *direct* for symmetries, it means orientation preserving.

At worst, I suppose you can get puzzles with 60 different pieces, all non-flat.

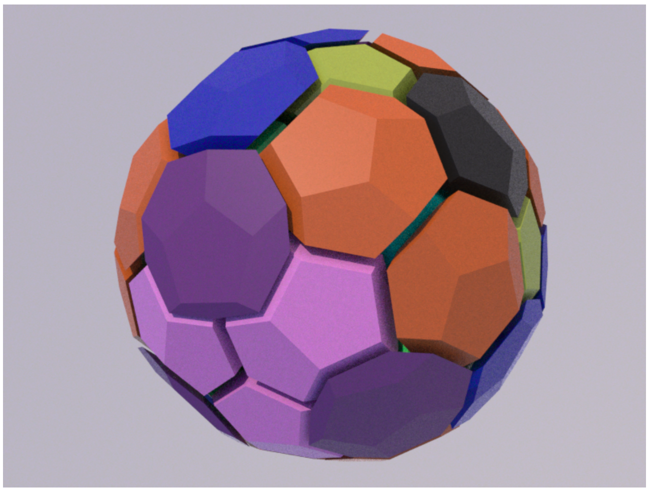


Vertex on top.

7 non-flat pieces types

1 flat type

Order 24 symmetry group (group of the regular tetrahedron). So 12 direct.

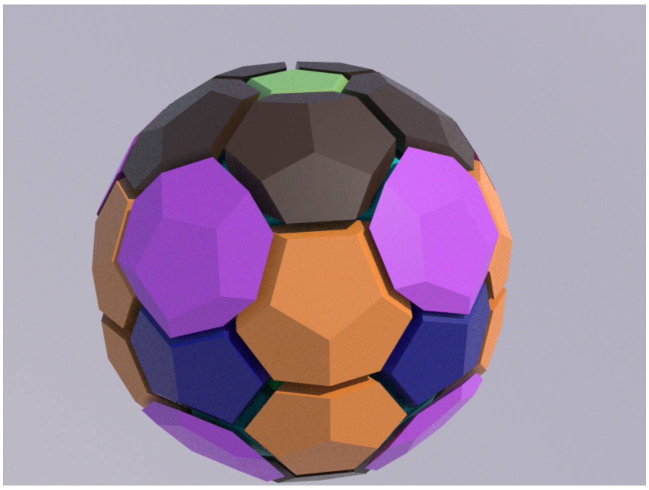


Edge mid-point on top.

10 non-flat types

2 flat types

Order 12 symmetry group (group of the regular triangle in space). So 6 direct.



Face centre on top.

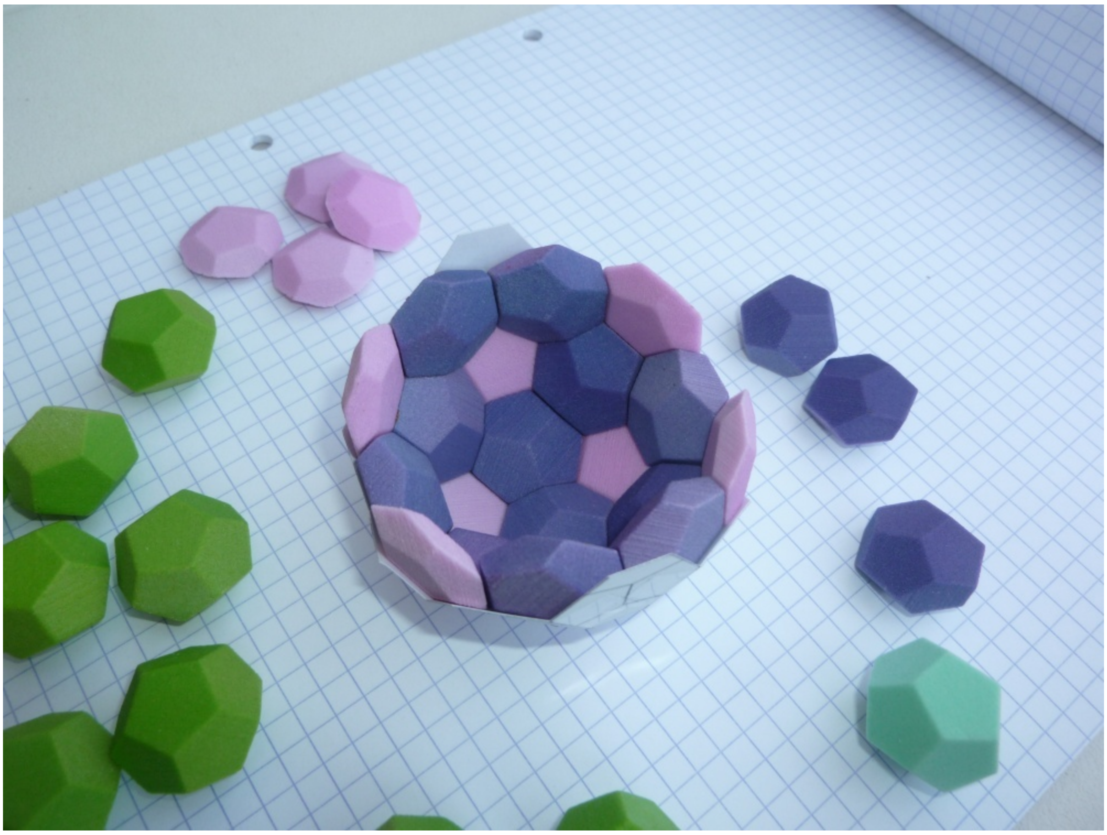
8 non-flat types

1 flat type

Order 20 symmetry group (group of the regular pentagon in space). So 10 direct.

Constructing the physical puzzle

I have tried two different 3D prints.



The first one was ordered on one of the main online 3d printing companies, not the cheapest. It was printed in coloured sandstone material, and the size was adjusted for a price around 80€ (not counting shipping).

I could have ordered 1, 12, 20 and 12 prints of each type, but I think it saves money to send instead an STL file where all the 45 pieces were in position and slightly displaced so that there is sufficient space between them (just multiply the centre position by some factor) for otherwise the pieces will fuse.

On this scale, the height of the central piece (distance between the planes supporting two opposite faces of the central dodecahedron) is approx 18mm.

The whole model as a height of approximately 57mm.

The stand is made of 15 identical Bristol board polygons glued together with transparent rubber tape. Finding the right scale for the polygons required several attempts because some unknown margin was to be added, and it is very sensitive: it has to be neither too loose or too tight.

Note that the upper half of the shell and the lower half have a different shape.

The second one was printed on a Makerbot Replicator II using PLA plastic. No support, no raft, filling: 10%.

The scale was chosen so that the height of the central piece (distance between the planes supporting two opposite faces of the dodecahedron) is 2 inches. The whole model has a height of approximately 16cm and weights around 1kg.

The overall printing process took 50h, not counting the failures. The absence of support may induce some deformation depending on the machine so you may have to add some (in fact the yellow pieces in the one I printed are slightly deformed but the defect was small enough). I presented the green and yellow faces so that they lay on their big face, the purple piece so that it lays on a small face.

It was then painted with acrylic paint and varnished for protection.

The 30 polygons of the shell were also 3D printed separately, laying flat, and then glued with a soldering iron and fine threads of PLA scraps into two half-shells, one consists in 15 polygons and the other one in 14. This bond is rather weak so there may be a better way. The last polygon was glued to two yellow pieces with cyanoacrylate glue: this is because those two pieces would not hold in place, there being non enough friction to prevent them from sliding. Note that the whole lower shell (or upper shell) would not fit in the printer so they really had to be printed piece by piece. Note also that the upper half of the shell and the lower half have a different shape, even without the hole on top.

Technical toolbox

Q: How do you orthogonally project from 4D to 3D?

A: Just map (x,y,z,t) to (x,y,z),

Q: How do you compute distances in 4D?

A: The distance from (x,y,z,t) to 0 is $\sqrt{x^2+y^2+z^2+t^2}$ and this is also the length of the vector v = (x,y,z,t). To get the distance between two points P=(x,y,z,t) and P'=(x',y',z',t'), compute their difference v=(x'–x, y'–y, z'–z, t'–t) and take its length.

Q: How do you rotate?

A: Rotations in 2D fixing the origin are given by multiplication by a rotation matrix: $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ where c = cos θ and s = sin θ. This means (x,y) $\rightarrow$ (cx – sy , sx + cy).

Rotations in 3D fixing the origin are more complicated but can be recovered as compositions of rotations along the 3 axes, which are much simpler: the coordinate along the axis is unchanged and the remaining two coordinates are transformed according to the same formula as for a 2D rotation. Be careful about the direction of rotation (orientation). To rotate by angle A along an axis passing through the origin, one may use the latitude and longitude coordinates of the intersection P of the rotation axis with a sphere centred on the origin. Then the isometry is: first rotate along the vertical axis by a quantity that will put back the longitude of P to 0. Then along the appropriate horizontal axis to put its latitude to 90. Then rotate by A. Then perform the inverse of the second rotation followed by the inverse of the first.

In any dimension, an orientation preserving isometry that fixes the origin is always a composition of simple rotations, which are defined as follows: they leave unchanged all but two coordinates, that they transform as above.

About

The author: my name is Arnaud Chéritat, I am a mathematician born in 1975, specialized in holomorphic dynamical systems, but very fond of geometry and topology.

I got much interested in polytopes when I was 18 years old.

Later, around 2010 I decided to try 3D printing, I chose a projected polytope and sent the files to the Sculpteo company. I added no magnets nor any mechanism to attach the pieces. When I received them, I realized I had no way to make them hold together, so I built a stand out of Bristol-type cardboard paper and, luckily enough, gravity and friction had the thing hold together.

In 2014, I 3D-printed them again in a much bigger version, in PLA with a Makerbot Replicator II, in the fablab CampusFab of my university (Université Paul Sabatier in Toulouse). I now bring this model around and show it to various people, from the very young to the adult.

Alba Málaga made paper models and used them successfully in *Salon Culture & Jeux Mathématiques 2016* in Paris.