DISCRETE GROUPS OF PLANE ISOMETRIES

A new classification and their representations as Wallpaper groups.

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Abstract:
This paper deals only with discrete groups containing two independent translations.
The introduction of the reduced group and a result about the reflection axes lead quickly to the list of all the possible frameworks (configurations of the reflection axes and the rotation's centers of maximum order); besides we get a simple and intuitive new notation.
It remains to verify the existence of these 17 groups with the use of tilings on which they act, and to make sure that none two of them are isomorphic.

Key words:
plane isometries, discrete groups, reflections, tilings

1 Introduction.

Experience shows that this subject often leads to misunderstandings which alter it completely. To be precise, our starting point is purely **algebraic**. Our aim was to determine "abstractly" all the possible structures for a discrete group of isometries of the plane.

After a quick reminder about the lattices and the euclidean group of the plane (§ 2) we:
1) first make an inventory of the 17 classes of all possible discrete groups of isometries containing two independent translations (§ 4),
2) then prove that each class actually exists (§ 5) using its action on a set of isometric tiles, all unmarked but one exception (see [Ma]),
3) finally establish the conjugation relations between groups of the same class and show that there is no isomorphism between two groups belonging to two different classes.

Works related to discrete groups often begin by studying the isometries of the lattice associated to the translations of such a group G. We believe that this method shows what G is not, instead of what it is.
Thus we introduced straightaway the **reduced** group of G (§ 3). There are two reasons for this choice:
1) A discrete group of isometries G does not always let the lattice associated to the translations of G invariant (example: the group g in § 4). On the other hand this property holds for the reduced group of G (see § 3, theorem 1).
2) The reduced group of G is isomorphic to the quotient of G by the sub-group of its translations, but it may happen that it is not isomorphic to none of the subgroups of G (see the end of § 4).

Using the complex plane **C** makes it easier to carry out the chosen method and to obtain the **frameworks** of the different groups.
The notations have been carefully chosen in such a way that the structures of the groups are suggested: the letters used are **c** (from "center"), **p** (from "pure"), **g** (from "glide") and **a** (from "alternate").

To remain as short as possible we failed to study the rosettes and the friezes.

2 Notations and generalities.

$\mathcal{I}, \mathcal{I}_E, \mathcal{T}, \mathcal{D}$ are the groups of the isometries of the plane, of the isometries which let a set E stable, of the translations, of the displacements and $\mathcal{I}^-$ is the set of the pure and glide reflections.
In the complex plane a translation t is identified with the affix of its vector, and $t+t'$ is the translation composed of t and t'.
Let $f(z)=az+b$ and $g(z)=cz+d$, where $a,b,c,d \in \mathbf{C}$, $|a|=|c|=1$, be two displacements; we call **distance** from f to g and we denote $\delta(f,g)$ the positive number

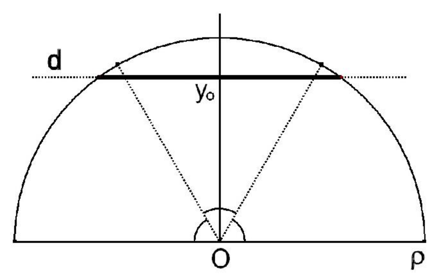
$$\max_{|z|=1} |f(z)-g(z)| = |a-c|+|b-d|$$

which defines a distance over the group $\mathcal{D}$.
The same holds for $f(z)=a\bar{z}+b$ and $g(z)=c\bar{z}+d$, belonging to $\mathcal{I}^-$.

Let A be a subset of a metric space (X,δ). A is said **discrete** when for any $a \in A$ there exists $r>0$ such that $B(a,r)\cap A=\{a\}$.
For that it is sufficient that for any ball B centered at a point of X the intersection $A\cap B$ is **finite.**
Conversely this condition is necessary if X is plunged in a numeric space $\mathbf{R}^n$ and A is closed in $\mathbf{R}^n$ because then $A\cap\overline{B}$ is discrete compact, thus finite.

Let $\mathcal{R}$ be a discrete **subgroup** of the group $(\mathbf{C},+)$. Then there exists $r>0$ such that any two distinct elements of $\mathcal{R}$ are at least at a distance r, thus $\mathcal{R}$ is closed. Using the characterization seen above let us show that $\mathcal{R}$ is generated by one or two elements.
"One" is obvious if $\mathcal{R}$ is reduced to 0 or included in a line (think about the euclidean division).

Otherwise consider $\rho\in\mathcal{R}$ with module >0 minimum and let us go back to the case where ρ itself is >0.
Let be y_0 real such that $0<|y_0|<\rho\dfrac{\sqrt{3}}{2}$, $d=\{z|\Im z=y_0\}$.
Then the chord $C=d\cap D(O,\rho)$ has a length $>2\rho\sqrt{1-\dfrac{3}{4}}=\rho$.
Suppose that $d\cap\mathcal{R}$ is not empty. Then there exists x_0 such that $\mathcal{R}\cap d$ may be written $x_0+iy_0+\rho\mathbf{Z}$ so that $C\cap\mathcal{R}$ would be non empty. This contradicts the definition of ρ.
For any $x+iy\in\mathcal{R}$, not real, we have $|y|\geq\rho\dfrac{\sqrt{3}}{2}$.



Knowing that $\mathcal{R}$ is discrete there exists $\rho'>0$, element of $\mathcal{R}$ with minimum imaginary part >0, and from that we deduce that $\mathcal{R}$ is generated by $\{\rho,\rho'\}$.

$\mathcal{R}$ is called a **lattice**. Examples: $\mathbf{Z}[i], \mathbf{Z}[j]$.

Let us close this section with two properties of the lattices:

1) Let $\mathcal{R}$ be a lattice generated by $\{\rho,\rho'\}$ with $0<\rho\leq|\rho'|$ like above. We suppose $\mathcal{R}$ stable by a rotation with center O and angle outside $\pi\mathbf{Z}$.
If α is the minimum acute angle of the rotations with center O which stabilize $\mathcal{R}$ we have $\sin\alpha\geq\dfrac{\sqrt{3}}{2}$ (see above) and $\alpha\geq\dfrac{\pi}{3}$. Hence $\alpha\in\left[\dfrac{\pi}{3},\dfrac{\pi}{2}\right]$.
On the other hand 2π must be multiple of α (euclidean division) but the value $\dfrac{2\pi}{5}$ must be excluded for it would imply a rotation with angle $\pi-\dfrac{4\pi}{5}<\dfrac{\pi}{3}$.

Finally we conclude that $\{2,4,6\}$ is the set of the possible values for card $[\mathcal{D}\cap\mathcal{I}_O\cap\mathcal{I}_\mathcal{R}]$.

2) Let $\mathcal{R}$ be a lattice stable by the reflection with axis Δ which we suppose be the real axis. Then $\mathcal{R}$ is also stable by the mapping $z\mapsto-\bar{z}$ which is the reflection about the imaginary axis. For any $z\in\mathcal{R}$, $z+\bar{z}$ and

$z-\bar{z}$ belong to $\mathcal{R}$ and respectively to each of the axes introduced above. Thus $\mathcal{R}$ contains z_0 real (resp. z_0' pure imaginary) with modules >0 minima.
If x and y are non integer reals such that $xz_0 + yz_0' \in \mathcal{R}$ we necessarily have $xz_0 - yz_0' \in \mathcal{R}$ and $2x$ and $2y \in \mathbf{Z}$ by addition and substraction of these elements of $\mathcal{R}$.
Conversely suppose that $2x$ and $2y$ are two odd integers. Then there exist m and n integers such that

$$xz_0 + yz_0' = mz_0 + nz_0' + \frac{z_0 + z_0'}{2}.$$

Finally $xz_0 + yz_0' \in \mathcal{R}$ if and only if $\dfrac{z_0 + z_0'}{2} \in \mathcal{R}$.

3 Reduced isometry and reduced group.

Let f and g be two isometries defined by $f(z) = az + b$ and $g(z) = c\bar{z} + d, (|a| = |c|) = 1)$. We set $\tilde{f}(z) = az$ and $\tilde{g}(z) = c\bar{z}$ and call these isometries **reduced** of f and g.
Examples: The reduced of a translation is the identity map, for a rotation it is a rotation with same angle, and for a reflection (pure or glide) it is a pure reflection with parallel axis.

If G is a group of isometries, a simple computation shows that we have defined a surjective morphism from G over a group called **reduced group** of G denoted $\tilde{G}$ (quotient group of G by its translations' subgroup).

Theorem 1: *Let G be a group of isometries, then the reduced group $\tilde{G}$ is a subgroup of $\mathcal{I}_O[(G \cap \mathcal{T})(O)]$ (isometries of the lattice associated to the translations).*

Proof: For $f(z) = az + b$ and $t \in G \cap \mathcal{T}$, we verify the relation $\tilde{f}[t(O)] = f \circ t \circ f^{-1}(O)$ and the same holds for $f \in G \cap \mathcal{I}^-$. Since $G \cap \mathcal{T}$ is a normal subgroup of G we are done.

Theorem 2: *The group of isometries G is* **discrete** *if and only if the set $(G \cap \mathcal{T})(O)$ is discrete.*

Proof: Only the part "if" needs to be proved. By hypothesis there exists $\rho > 0$ so that two distinct elements of $(G \cap \mathcal{T})(O)$ are at least distant by ρ. Let us first show the analogous for $G \cap \mathcal{D}$.
Let be $f(z) = az + b$ and $g(z) = cz + d$. If $a = c$ we have $\delta(f, g) = |b - d|$; now $f \circ g^{-1}$ is precisely the translation whose vector has $b - d$ as affix; this translation belongs to G and we have $|b - d| \geq \rho$.
Let now be $a \neq c$ and $t \in G \cap \mathcal{T}$ such that $|t(O)| = \rho$, then we have

$$\delta(f, g) \geq |a - c| = \frac{1}{\rho}|at(O) - ct(O)| = \frac{1}{\rho}|f \circ t \circ f^{-1}(O) - g \circ t \circ g^{-1}(O)| \geq 1.$$

Finally two distinct elements of $G \cap \mathcal{D}$ are distant by at least $\min(\rho, 1)$. The same proof shows that, if it is not empty, $G \cap \mathcal{I}^-$ is discrete. Both together prove that G itself is discrete (see the characterization at the beginning of § 2).
For every integer $n > 0$ we denote $\mathbf{C_n}$ the cyclic group built by the rotations with center O and angle in $\dfrac{2\pi}{n}\mathbf{Z}$, and $\mathbf{D_n}$ a $2n$ order dihedral group which contains $\mathbf{C_n}$ (that is the group of the isometries which fix an n vertices regular polygon centered at O).

Theorem 3: *Let G be a discrete group of isometries, then the reduced group $\tilde{G}$ is one of the following ten subgroups of $\mathcal{I}_O$:* $\mathbf{C_1}, \mathbf{D_1}, \mathbf{C_2}, \mathbf{D_2}, \mathbf{C_3}, \mathbf{D_3}, \mathbf{C_4}, \mathbf{D_4}, \mathbf{C_6}, \mathbf{D_6}$.

Proof: Indeed by theorem 1 we know that $\tilde{G}$ is included in the group of isometries of $(G \cap \mathcal{T})(O)$ and at § 2 we have seen that for this group the only possible rotations' angles are 0 or the multiples of $\dfrac{\pi}{3}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}$ or π.

Remark: Notice that the groups $\mathbf{C_2}$ and $\mathbf{D_1}$ are algebraically isomorphic but contain isometries of different types.
Notations: We call **degree** of the discrete group G the order of the group $\tilde{G} \cap \mathcal{D}$ and we denote $\mathcal{R}$ the lattice $(G \cap \mathcal{T})(O)$.

Theorem 4: *Let G be a discrete displacements' group of degree $d \geq 2$.*
Then, taking the origin O at a rotation center of order d, the lattice $\mathcal{C}_d$ of these centers is linked to the lattice $\mathcal{R}$ associated to the translations of G by the appropriate relation among:

$$\mathcal{C}_2 = \frac{1}{2}\mathcal{R} \qquad \mathcal{C}_3 = \frac{1-j}{3}\mathcal{R} \qquad \mathcal{C}_4 = \frac{1+i}{2}\mathcal{R} \qquad \mathcal{C}_6 = \mathcal{R}.$$

Remark: All these lattices are symmetrical with respect to the origin.
Proof: It is sufficient to compose the rotations of order d and center O with the translations of G.
Example for $d = 3$: Let r be the image of O by a translation, then $z \mapsto z + r \mapsto e^{i\frac{2\pi}{3}}(z + r) = j(z + r)$ is a rotation of order 3 whose center is defined by $z = j(z + r)$, that is $z = \dfrac{j}{1-j}r = \dfrac{j(1-j^2)}{(1-j)(1-j^2)}r = \dfrac{j-1}{3}r$.

4 Axes of a discrete group of isometries. Classification.

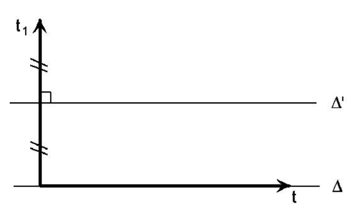
Let G be a discrete group of isometries, Δ_1 a line through O, σ_1 the pure reflexion with axis Δ_1.
Then $\sigma_1 \in \tilde{G}$ if and only if there exists in G a reflection (pure or glide) with axis Δ parallel to Δ_1. We then say that Δ is an **axis** of G.
If any reflexion $\in G$ with axis Δ is a glide reflexion we say that Δ is a **glide** axis of G.
If G contains the pure reflection with axis Δ we say that Δ is a **pure** axis of G; let us notice that G contains also glide reflections with axis Δ because G contains translations of direction Δ (see §2).

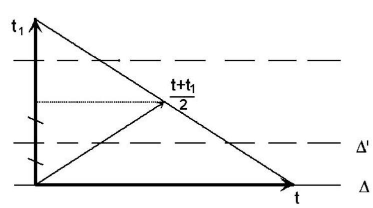
Let Δ be an axis of G, σ the pure reflection with axis Δ, t (resp. t_1) a non null translation of G, minimal with direction Δ (resp. perpendicular to Δ). There exists m integer such that $\sigma \circ m\dfrac{t}{2} \in G$.

Lemma: *Let α be a real, $\Delta' = \alpha t_1(\Delta)$ (parallel to Δ). Then Δ' is an axis of G if and only if there exists an integer m' such that $2\alpha t_1 + (m' - m)\dfrac{t}{2} \in G$.*

Proof: Let $\sigma' = 2\alpha t_1 \circ \sigma$ be the pure reflection with axis Δ', and let m' be an integer.
Then $\quad \sigma' \circ m'\dfrac{t}{2} = 2\alpha t_1 \circ \sigma \circ m'\dfrac{t}{2} = 2\alpha t_1 \circ \left(\sigma \circ m\dfrac{t}{2}\right) \circ (m' - m)\dfrac{t}{2}$, and we are done.

Theorem 5: *Let G be a discrete group of isometries, Δ an axis of G, and t (resp. t_1) a non null minimal translation of G with direction Δ (resp. perpendicular to Δ). Then*
1) If $\dfrac{t + t_1}{2} \notin G$ all the axes of G parallel to Δ are of the same kind as Δ and two by two distant by an element of $\dfrac{|t_1|}{2}\mathbf{N}^*$.
2) If $\dfrac{t + t_1}{2} \in G$ there exists in G a second family of axes parallel to Δ, all of different kind of the preceding ones, and at mid distance between them.





Proof: 1) Using the lemma and the end of § 2, Δ' is an axis of G if and only if:
a) $2\alpha t_1 \in G \iff 2\alpha$ is an integer which gives the position of Δ' relatively to Δ,
and b) $(m' - m)$ is even $\iff \Delta'$ is an axis of the same kind as Δ.

2) Now we have to look at the case $2\alpha \notin \mathbf{Z}$. Δ' is an axis of G if and only if:
a) $2\alpha - \dfrac{1}{2} \in \mathbf{Z} \iff 4\alpha$ is odd. Hence the position of Δ',
and b) $\dfrac{m' - m}{2} \notin \mathbf{Z} \iff (m' - m)$ is odd $\iff \Delta$ and Δ' are of different kinds.

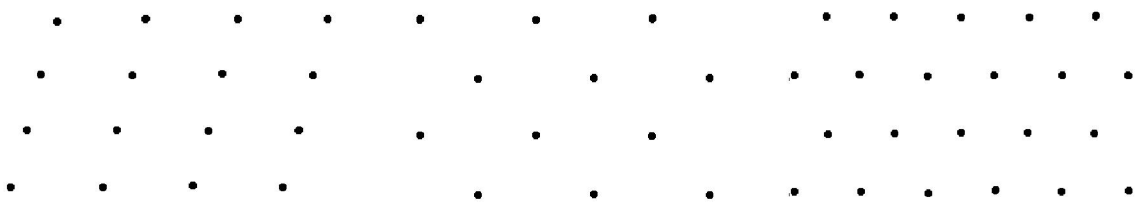
Theorem 1 established an unilateral link between the symmetries of $\mathcal{R}$ and the directions of the reflection axes of G. Theorem 5 clarifies the position of theses axes.
Let us notice that if the axes of two pure reflections of G intersect at a point O, these reflections generate a dihedral subgroup of G, also included in the reduced group (with fixed point O).

Theorem 6: *Let G be a discrete group of isometries, H a dihedral subgroup with fixed point O, $\mathcal{R}$ the lattice of the images of O by the translations of G. Let σ be a pure reflection with axis parallel to the axis of a reflection σ' of H. Then σ belongs to G if and only if σ stabilizes $\mathcal{R}$.*

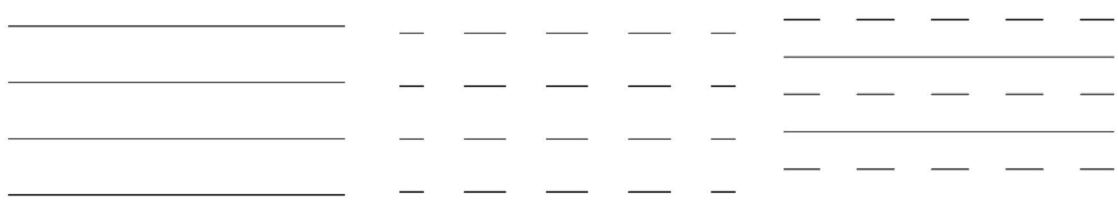
Proof: Let u be the translation $\sigma \circ \sigma'$.
Knowing $\sigma'(\mathcal{R}) \subset \mathcal{R}$ (th. 1), we have: $\sigma(\mathcal{R}) \subset \mathcal{R} \iff u(\mathcal{R}) \subset \mathcal{R} \iff u \in G$,
and knowing $\sigma' \in G$ we have: $u \in G \iff \sigma \in G$.

Consequences: Let G be a discrete group of isometries of degree d. We call **framework** of G the representation of the pure axes (continuous lines), of the glide axes (dotted lines), and of the rotation's centers of order d if $d \geq 2$ (points).

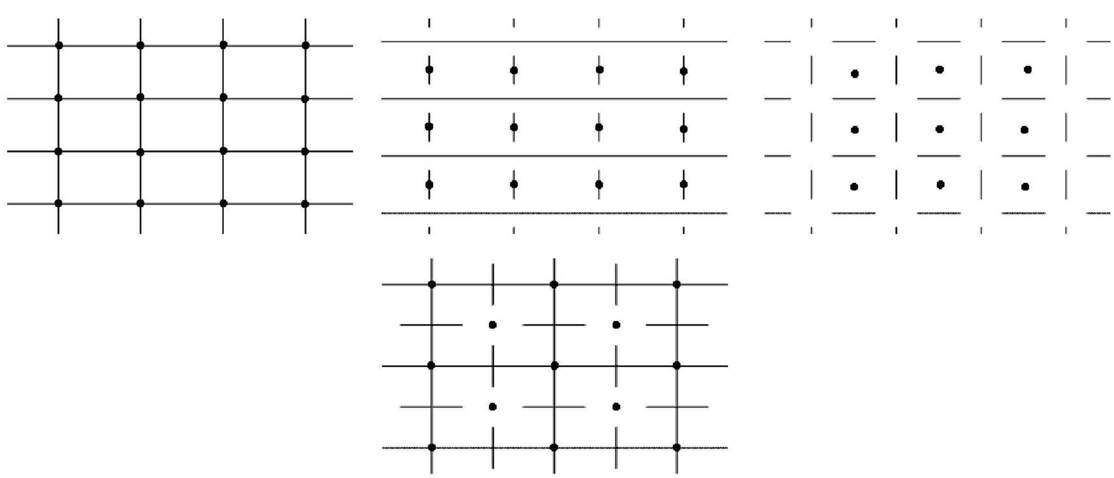
If G does not contain reflections, there are five cases denoted $\mathbf{c_1}$ (neither rotations nor reflections), $\mathbf{c_2}$ (left framework), $\mathbf{c_4}$ (middle framework), $\mathbf{c_3}$ and $\mathbf{c_6}$ (identical frameworks, on the right).



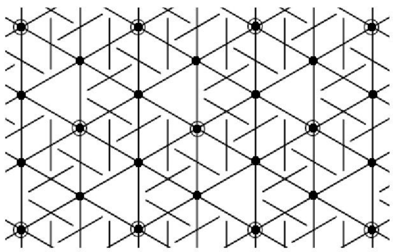
If $d = 1$ the axes of G are parallel, all pure or all glided, or alternatively pure and glided when $\mathcal{T} \cap G$ verifies the **condition of alternation** $\dfrac{t + t_1}{2} \in G$. Thus three cases denoted **p**, **g** and **a**.

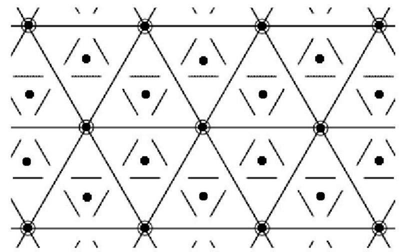


If $d = 2$ we have to envisage axes in two perpendicular directions for which the condition of alternation is verified (or not), simultaneously by symmetry of the role of Δ and $\Delta^{\perp}$, which eliminates the combinations *pa* and *ga*. Thus the four associations $\mathbf{p^2}$, $\mathbf{pg}$, $\mathbf{g^2}$ and $\mathbf{a^2}$.

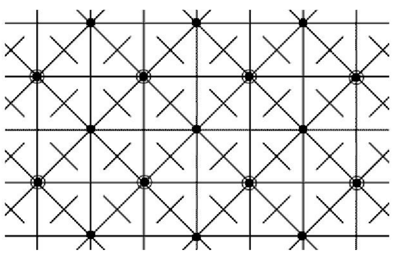


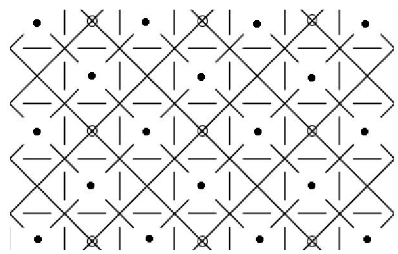
If $d = 3$ the lattice $\mathcal{R}$ must be symmetrical with respect to the directions of three axes with the polar angles $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ (see theorem 1), so the points of $\mathcal{R}$ are the vertices of equilateral triangles which pave the plane. Furthermore the three directions of symmetry verify the condition of alternation. Hence there exists in G pure axes which intersect at a point noted Ω. If $\mathcal{R}_\Omega = (G \cap \mathcal{T})(\Omega)$ then by theorem 6 the pure axes of G may be either the bisectors of the triangles associated to $\mathcal{R}_\Omega$ or the sides of these triangles. We denote the corresponding groups by $\mathbf{a^3}$ and $\mathbf{a_c^3}$.
Remark: Note that the triangles above are those defined by the lattice $\mathcal{R}_\Omega$ which is different from the set of rotation's centers of maximum order (the points on the framework). To clarify this difference, the points of $\mathcal{R}_\Omega$ are indicated with small circles on the next four diagrams below.



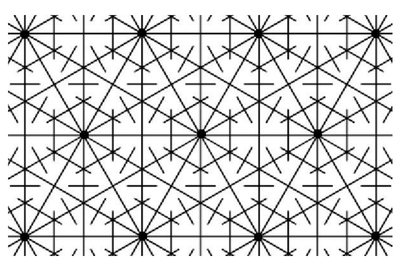


If $d = 4$ the lattice $\mathcal{R}$ must be symmetrical with respect to the directions of four axes with the polar angles $0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}$ (see theorem 1), so the points of $\mathcal{R}$ are the vertices of squares which pave the plane.
Furthermore among the four directions of symmetry only two verify the condition of alternation. This suffices in view to find in G two pure axes which intersect at a point noted Ω. If $\mathcal{R}_\Omega = (G \cap \mathcal{T})(\Omega)$ then by theorem 6 the pure axes parallel to the diagonals of the squares associated to $\mathcal{R}_\Omega$ are these diagonals themselves and if they exist the pure axes parallel to the sides of these squares are these sides themselves and the bisectors of the sides; in this last case we write $\mathbf{p^2a^2}$. In the other case we write $\mathbf{g^2a^2}$.





If $d = 6$ the lattice is the same as for $d = 3$ but we have to combine the two families of axes. Thus only one possibility: $\mathbf{a^6}$.



Remarks: 1) One may have noticed the efficiency of the reduced group as classification tool of the discrete groups. In comparison it seems difficult to undertake this task starting with the lattice $\mathcal{R}$, all the more to search a priori a base of it.
2) Let G be the group of the isometries $z \mapsto z + 2m + pi$ and $z \mapsto \bar{z} + 2n + 1 + qi$, where m, n, p, q are integers (group denoted g above). The reduced group is not isomorphic to any subgroup of G, which itself does not let the lattice $\{2m + pi\}$ invariant.

5 Existence of the discrete groups of isometries. Generators

The aim of the **representation** of the 17 discrete groups as groups of isometries of 17 types of tilings (see last page) is to prove the existence of these groups; only the necessary properties have been yet proved.

The 17 groups enumerated above each act on one of these sets of isometric tiles. Note that one tile is "decorated"; indeed according to [Ma] and [G.S.] the corresponding group cannot be associated to a tiling with "naked" tiles.
From left to right and top to bottom the group acting on these sets of tiles are:

$$c_2 \quad g^2a^2 \quad a_c^3 \qquad c_1 \quad c_6 \quad p^2 \qquad p^2a^2 \quad a \quad g^2 \qquad a^2 \quad g \quad c_4 \qquad a^6 \quad a^3 \quad p \qquad c_3 \quad pg$$

Summary of the 10 reduced groups with their 17 associated isometries' groups.

$\mathbf{C_1}$	$\mathbf{D_1}$	$\mathbf{C_2}$	$\mathbf{D_2}$	$\mathbf{C_3}$	$\mathbf{D_3}$	$\mathbf{C_4}$	$\mathbf{D_4}$	$\mathbf{C_6}$	$\mathbf{D_6}$
$\mathbf{c_1}$	**p g a**	$\mathbf{c_2}$	$\mathbf{p^2}$ **pg** $\mathbf{g^2}$ $\mathbf{a^2}$	$\mathbf{c_3}$	$\mathbf{a^3}$ $\mathbf{a_c^3}$	$\mathbf{c_4}$	$\mathbf{p^2a^2}$ $\mathbf{g^2a^2}$	$\mathbf{c_6}$	$\mathbf{a^6}$
no rotations		only half-turns		only third of turns		quarter turns		sixth of turns	

Remarks: One may study the generators of the discrete groups of isometries, in particular if, for G not contained in $\mathcal{D}$, the reflections of G generate G. Using $G \cap \mathcal{T}$, one finds (see [Bo]) that the answer is yes, but for the group p. In this case all the axes have the same direction and all the reflections are pure so that they generate only a frieze group isomorph to the one generated by two half-turns.
One may also wonder if the **position** of the axes of G is sufficient to determine G. It still holds for the groups p, a, g, a_c^3, a^3 (degrees 1 and 3).
Finally, if G of degree d, is contained in $\mathcal{D}$, then G is generated by the rotations of order d, and even by only two among them for $d > 2$.

6 Isomorphisms and conjugations.

Theorem 7: *There is no isomorphism between two of the 17 groups above.*

We have to prove several lemmas bringing into play the **commutator** of a part E of G, namely the set of the elements of G which commute with every element of E. We denote this set $\Gamma(E)$.
Furthermore let us remind that we dismissed the study of the rosettes and friezes, thus $G \cap \mathcal{T}$ is not generated by only one element.

Lemma 1: *Let Φ be an isomorphism from the discrete group G to the discrete group G_1. Then Φ maps $G \cap \mathcal{T}$ on $G_1 \cap \mathcal{T}$.*

Let us suppose that this does not happen. Then it would exist in G for instance a translation t such that $\Phi(t)$ would not be a translation; $\Phi(t)$, being not of finite order, would be a glide reflection which we denote g. But Φ maps $\Gamma(t)$ onto $\Gamma(g)$. Let us distinguish two cases:

a) The two commutators would contain some elements of order 2. Then $\Gamma(g)$ would be commutative while $\Gamma(t)$ would not be. This is impossible.
b) The two commutators would not contain elements of order 2. Then $\Gamma(g)$ would be generated by only one element and $\Gamma(t)$ would not be. This is impossible again.
The set of translations is algebraically stable and consequently for two isomorphic groups G and G_1 their reduced groups $\tilde{G}$ and $\tilde{G}_1$ are isomorphic.

Now we may consider two discrete groups whose reduced groups are isomorphic.

Remark: Before the complete proof of theorem 7 let us give an algebraic characterization for the notions of pure and glide axes:

a) In a discrete group there are pure axes if and only if there are elements of order 2 whose commutator is infinite.
b) In a discrete group there are glide axes if and only if there are elements which are not of finite order and whose commutator is generated by only one element.

Lemma 2: *If $\tilde{G}$ is a copy of $\mathbf{C_2}$ and $\tilde{G_1}$ a copy of $\mathbf{D_1}$ then G and G_1 are not isomorphic.*

Necessarily we have $G = c_2$ which has no axes and $G_1 = g, p$ or a which have some axes. The existence of axes being **algebraic** G and G_1 are not isomorphic.

Lemma 3: *For the following couples no isomorphism is possible because the first group has pure axes but the second one has none:* (p, g) (a, g) (p^2, g^2) (pg, g^2).

Lemma 4: *For the following couples no isomorphism is possible because the first group has glide axes but the second one has none:* (a, p) (a^2, p^2) (pg, p^2).

For the couple (a_c^3, a^3) only the first group contains subgroups $\mathbf{C_3}$ not included in a subgroup $\mathbf{D_3}$.
For (a^2, pg) (resp. (p^2a^2, g^2a^2)) only the first group contains subgroups $\mathbf{D_2}$ (resp. $\mathbf{D_4}$).

Theorem 8: *Two copies of a same discrete group of isometries of degree d are conjugated*
a) by a similitude if $d \geq 3$,
b) by a similitude composed with an orthogonal affinity if $d \leq 2$ and G not included in $\mathcal{D}$,
c) by the composition of a similitude, an orthogonal affinity and a transvection if $d \leq 2$ and $G \subset \mathcal{D}$.

Proof: For any affine map ϕ, associating the framework and the lattice of G with those of G_1, we have $G_1 = \phi \circ G \circ \phi^{-1}$. In case a) the frameworks and the lattices are similar, in case b) a rectangle has to be mapped on an other rectangle, and in case c) a parallelogram has to be mapped on an other parallelogram.

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